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Details of the Derivation of the Acoustic Surface Loss Formula

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ABSTRACT

The purpose of this document is to provide a detailed and self-contained derivation of the high frequency component SL_1 of the acoustic surface loss formula for a rough interface of impedance mismatch. We thoroughly examine the existing derivations of SL_1, point at their flaws and suggest the corrected form.

Much of the report is a technical review in which we critically revisit the fundamentals behind the surface loss formula and cover such aspects as the basics of the Kirchhoff theory, random surfaces and wave scattering from them, large roughness approximation of the diffuse field, and energy flux calculations.

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EXECUTIVE SUMMARY

Tactical decisions in ASW often have to be based on projected performance of sonars. Sonar performance prediction in turn strongly relies on acoustic propagation modelling. A contemporary model for underwater sound propagation is typically a complex software which implements modules describing such phenomena as

- acoustic propagation in an ocean with a general sound velocity profile,
- reflection of sound from ocean's bottom and surface (surface and bottom loss sub-models),
- volume, bottom and surface scattering of sound and the resulting reverberation.

It is often the case that the model provides more than one sub-model for the same phenomenon and the user has to decide which of them is most appropriate for the task. Understanding the fundamentals upon which various sub-models are based helps the modeller to make more informative decisions on their suitability.

In this work we revisit the derivations of the surface loss formula used by the "Beckmann-Spizzichino" model included in a number of sonar performance prediction tools. Our resultant expression for the high-frequency surface loss term SL_1 is in variance from that currently employed by the model. This result suggests that in the sonar performance modelling we should use a newly-obtained expression for SL_1 or other available surface loss models.

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Appendices

1 Introduction

The high frequency component SL_1 of the acoustic surface loss formula [2, eqn. 3-8, p. 18] broadly used in acoustic propagation modelling and cited in the literature (e.g. [1], [2], [3], [4]), is obtained in a number of steps and has a deceptively compact and simple form

$$\mathrm{SL}_1 = -10\log_{10}(1-V), \tag{1}$$

where V is given by (96) and depends on the angle of the incident field, the surface correlation length and the RMS value of the surface elevation with respect to the mean plane.

The term SL_1 is often referred to as the "Beckmann-Spizzichino" large-roughness surface loss, presumably because some elements of its derivation are indeed based on a large-roughness approximation of the diffuse component of the reflected field obtained in [6, eqn. 47, p. 87]. However, the authorship of SL_1 can hardly be attributed to the authors of [6]. Reports [2, 1] relate the origins of the SL_1-term to the documentation for the RAYMODE X propagation model, but provide very sketchy demonstrations.

Because the derivations of SL_1 available in the mainstream literature contain errors, gaps and typographical mistakes, the user's confidence in the final result is undermined. The purpose of this work is to critically revisit the fundamentals behind the surface loss formula and produce a document with its detailed and self-contained derivation.

The structure of this document is as follows:

- In Section 2, largely based on [5], we review the Kirchhoff theory and express the complex amplitude of the scattered field in terms of the integral over the mean plane.
- In Section 3 we consider the coherent and diffuse fields and the associated intensities and examine the derivations provided in [6] and [5].
- In Section 4 we consider the asymptotic behaviour of the ensemble average of the diffuse intensity at large values of the roughness parameter and confirm the key result ([6, eqn. (47), p. 87]) required for derivation of the surface loss term SL_1. We also estimate the remainder of the resulting large-roughness approximation of the diffuse field.
- In Section 5 we carry out energy flux calculations. We estimate the total flow of the diffuse energy and use the result to derive the surface loss formula. Step-by-step analysis of the derivation allows us to identify the flaws in the existing version of the SL_1 term and suggest an alternative expression.

2 Complex amplitudes of scattered fields

The complex amplitude of an acoustic field (scaled pressure) resulted from scattering of an incident field $\psi^{\mathrm{inc}}(\mathbf{r})$ by a surface S_0 enclosing a bounded volume satisfies the integral relation

$$\psi^{\mathrm{sc}}(\mathbf{r}) = \int_{S_0}\left[\frac{\partial\psi(\mathbf{r}_0)}{\partial n_0}G(\mathbf{r}-\mathbf{r}_0) - \psi(\mathbf{r}_0)\frac{\partial G(\mathbf{r}-\mathbf{r}_0)}{\partial n_0}\right]dS_0 \tag{2}$$

where $\psi(\mathbf{r}) \equiv \psi^{\rm inc}(\mathbf{r}) + \psi^{\rm sc}(\mathbf{r})$ is the total field, $\mathbf{r}_0$ is the integration variable, dS_0 is the surface area element at $\mathbf{r}_0$, $\partial/\partial n_0$ denotes the directional derivative along the normal $\mathbf{n}_0$ to S_0 at $\mathbf{r}_0$ (see Figure 1), the function $G(\mathbf{r})$ is given by

$$G(\mathbf{r}) \equiv -\frac{e^{ik|\mathbf{r}|}}{4\pi|\mathbf{r}|}\,, \tag{3}$$

and k is the wave number. Formula (2) is referred to in [5] as the Helmholtz integral and its derivation is based on the Gauss divergence theorem and the Green's formulae. The following incident fields are typically considered

$$\psi^{\rm inc}(\mathbf{r}) = e^{i\mathbf{k}_{\rm inc}\cdot\mathbf{r}} \qquad \text{(plane wave with the wave vector } \mathbf{k}_{\rm inc}\text{)}, \tag{4}$$

$$\psi^{\rm inc}(\mathbf{r}) = G(\mathbf{r}-\mathbf{q}) \qquad \text{(field from a point source located at } \mathbf{q}\text{)}, \tag{5}$$

where $\mathbf{q}$ is not on the surface S_0.

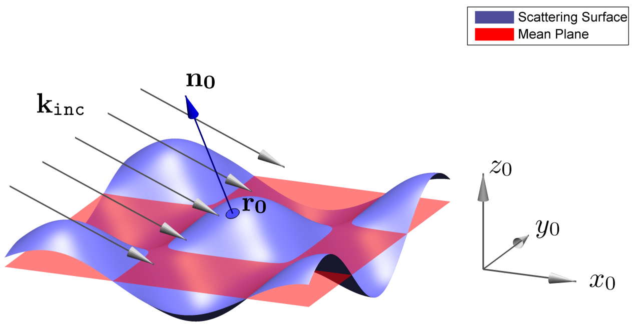

Figure 1: *The scattering surface S_0, the mean plane, the radius-vector $\mathbf{r}_0$, and vectors $\mathbf{k}_{inc}$ and $\mathbf{n}_0$. The acoustic field is considered in the volume $z_0 > h(x_0, y_0)$, where $z_0 = h(x_0, y_0)$ is the functional form of S_0.*

The proofs of the equation (2) are different for (4) and (5), because the behaviour of these two functions is different – the field (4) is regular everywhere, but does not satisfy the Sommerfeld radiation condition [7, eqn. (9A-12), p. 355], whereas the field (5) satisfies the Sommerfeld radiation condition, but is singular at $\mathbf{q}$. We revisit the derivation of the Helmholtz integral for the cases (4) and (5) in Appendix B.

As in [5] the incident field is assumed to be the plane wave (4). The Kirchhoff approximation adopts the assumption that the values and normal derivatives of the incident and reflected fields at the boundary are connected in the same way as in the case of reflection of a plane wave from a flat surface

$$\psi^{\rm sc}(\mathbf{r}_0) = R_0(\mathbf{r}_0)\psi^{\rm inc}(\mathbf{r}_0) \quad \text{and} \quad \frac{\partial\psi^{\rm sc}(\mathbf{r}_0)}{\partial n_0} = -R_0(\mathbf{r}_0)\frac{\partial\psi^{\rm inc}(\mathbf{r}_0)}{\partial n_0}\,, \tag{6}$$

where $R(\mathbf{r}_0)$ is the reflection coefficient at $\mathbf{r}_0$. Because, for the plane wave (4), we also have

$$\frac{\partial\psi^{\rm inc}(\mathbf{r}_0)}{\partial n_0} = i(\mathbf{k}_{\rm inc}\cdot\mathbf{n}_0)\psi^{\rm inc}(\mathbf{r}_0)\,,$$

we obtain

$$\psi(\mathbf{r}_0) = [1 + R_0(\mathbf{r}_0)]\psi^{\mathrm{inc}}(\mathbf{r}_0)\,, \tag{7}$$

$$\frac{\partial\psi(\mathbf{r}_0)}{\partial n_0} = i[1 - R_0(\mathbf{r}_0)](\mathbf{k}_{\mathrm{inc}} \cdot \mathbf{n}_0)\psi^{\mathrm{inc}}(\mathbf{r}_0)\,. \tag{8}$$

Application of (7)–(8) implies that $\mathbf{r}_0$ is not *shadowed* by other parts of the scattering surface, and throughout this report the shadowing effects are neglected. Substitution of (7) and (8) into (2) expresses $\psi^{\mathrm{sc}}(\mathbf{r})$ in terms of ψ^{inc}.

Note 1. The Fresnel reflection coefficient used as $R_0(\mathbf{r}_0)$ depends on the angle between $\mathbf{k}_{\mathrm{inc}}$ and $\mathbf{n}_0$ (see Figure 1). Since $\mathbf{n}_0$ depends on $\mathbf{r}_0$, the reflection coefficient also depends on $\mathbf{r}_0$, even if the reflecting material is the same throughout S_0. However, if S_0 is the sea surface, which is the case in our considerations, the Fresnel reflection coefficient can be assumed to be -1.

From now on S_0 will denote a patch of ensonified surface. This patch is assumed to be centered at the origin and its size $d \equiv \max_{\mathbf{r}_0 \in S_0}(r_0)$ to be much greater than the wavelength. It is also assumed that $r \gg d$ and that $kd^2/r \ll 2\pi$ (far-field approximation). These assumptions allow us to simplify the phase in $G(\mathbf{r} - \mathbf{r}_0)$

$$k|\mathbf{r} - \mathbf{r}_0| = k(r^2 - 2\mathbf{r} \cdot \mathbf{r}_0 + r_0^2)^{1/2} = kr - \mathbf{k}_{\mathrm{sc}} \cdot \mathbf{r}_0 + O(kr_0^2/r) \approx kr - \mathbf{k}_{\mathrm{sc}} \cdot \mathbf{r}_0\,, \tag{9}$$

where $\mathbf{k}_{\mathrm{sc}} \equiv k\mathbf{r}/r$, and to reduce $G(\mathbf{r} - \mathbf{r}_0)$ and its normal derivative to

$$G(\mathbf{r} - \mathbf{r}_0) \approx -\frac{e^{ikr - i\mathbf{k}_{\mathrm{sc}} \cdot \mathbf{r}_0}}{4\pi r} \tag{10}$$

$$\begin{aligned} \frac{\partial}{\partial n_0} G(\mathbf{r} - \mathbf{r}_0) &= -\frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{4\pi|\mathbf{r} - \mathbf{r}_0|}\left(ik - \frac{1}{|\mathbf{r} - \mathbf{r}_0|}\right)\frac{\partial|\mathbf{r} - \mathbf{r}_0|}{\partial n_0} \\ &= \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{4\pi|\mathbf{r} - \mathbf{r}_0|}\left(ik - \frac{1}{|\mathbf{r} - \mathbf{r}_0|}\right)\frac{(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n}_0}{|\mathbf{r} - \mathbf{r}_0|} \\ &\approx \frac{e^{ikr - i\mathbf{k}_{\mathrm{sc}} \cdot \mathbf{r}_0}}{4\pi r}(\mathbf{n}_0 \cdot \mathbf{k}_{\mathrm{sc}})i\,. \end{aligned} \tag{11}$$

Substitute (7), (8), (10) and (11) into (2) to obtain

$$\psi^{\mathrm{sc}}(\mathbf{r}) = \frac{ie^{ikr}}{4\pi r}\int_{S_0}\left[(R_0\mathbf{k}^- - \mathbf{k}^+) \cdot \mathbf{n}_0\right]e^{i\mathbf{k}^- \cdot \mathbf{r}_0}dS_0\,, \tag{12}$$

where $\mathbf{k}^- \equiv \mathbf{k}_{\mathrm{inc}} - \mathbf{k}_{\mathrm{sc}}$ and $\mathbf{k}^+ = \mathbf{k}_{\mathrm{inc}} + \mathbf{k}_{\mathrm{sc}}$.

Let x_0, y_0 and z_0 be Cartesian coordinates induced by the mean plane as shown in Figure 1 and let $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ be the unit vectors of the respective axes.

Figure 2 introduces the angles θ_1, θ_2 and θ_3 which describe the directions associated with the incident and scattered fields. Note that the vector $\mathbf{k}_{\mathrm{sc}}$ depends on the radius-vector $\mathbf{r}$ of the observation point.

If the functional form of the surface S_0 is $z_0 = h(x_0, y_0)$ and S_M is the projection of S_0 on the mean plane, then

$$\mathbf{n}_0 = \frac{-\mathbf{e}_x\frac{\partial h}{\partial x_0} - \mathbf{e}_y\frac{\partial h}{\partial y_0} + \mathbf{e}_z}{\sqrt{\left(\frac{\partial h}{\partial x_0}\right)^2 + \left(\frac{\partial h}{\partial y_0}\right)^2 + 1}} \tag{13}$$

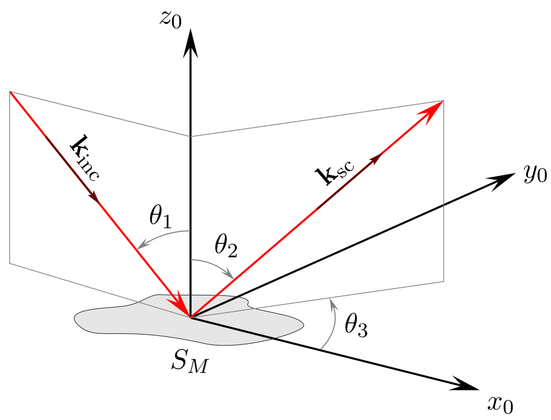

Figure 2: *Angles θ_1, θ_2 and θ_3.*

$$dS_M = dS_0(\mathbf{n}_0 \cdot \mathbf{e}_z) = \frac{dS_0}{\sqrt{\left(\frac{\partial h}{\partial x_0}\right)^2 + \left(\frac{\partial h}{\partial y_0}\right)^2 + 1}} \tag{14}$$

the integral (12) can be reduced to the integral over S_M

$$\psi^{\rm sc}(\mathbf{r}) = \frac{ie^{ikr}}{4\pi r} \int_{S_M} \left[(R_0\mathbf{k}^- - \mathbf{k}^+) \cdot (-\mathbf{e}_x h_{x_0} - \mathbf{e}_y h_{y_0} + \mathbf{e}_z)\right] e^{i\mathbf{k}^- \cdot \mathbf{r}_0} dx_0 dy_0 \tag{15}$$

Substitute

$$\mathbf{k}_{\rm inc} = k(\sin\theta_1 \mathbf{e}_x - \cos\theta_1 \mathbf{e}_z) \tag{16}$$
$$\mathbf{k}_{\rm sc} = k(\sin\theta_2 \cos\theta_3 \mathbf{e}_x + \sin\theta_2 \sin\theta_3 \mathbf{e}_y + \cos\theta_2 \mathbf{e}_z) \tag{17}$$

into (15) to obtain

$$\psi^{\rm sc}(\mathbf{r}) = \frac{ike^{ikr}}{4\pi r} \int_{S_M} \left(a\frac{\partial h}{\partial x_0} + b\frac{\partial h}{\partial y_0} - c\right) \times \exp\left\{ik\left[Ax_0 + By_0 + Ch(x_0, y_0)\right]\right\} dx_0 dy_0 \tag{18}$$

where

$$A = \sin\theta_1 - \sin\theta_2 \cos\theta_3 \tag{19}$$
$$B = -\sin\theta_2 \sin\theta_3 \tag{20}$$
$$C = -(\cos\theta_1 + \cos\theta_2) \tag{21}$$

and

$$a = \sin\theta_1(1 - R_0) + \sin\theta_2 \cos\theta_3(1 + R_0) \tag{22}$$
$$b = \sin\theta_2 \sin\theta_3(1 + R_0) \tag{23}$$
$$c = \cos\theta_2(1 + R_0) - \cos\theta_1(1 - R_0) \tag{24}$$

The coefficients A, B and C are the x, y and z components of the vector $\mathbf{k}^-/k$, while the coefficients a, b and c are the x, y and z components of the vector $(\mathbf{k}^+ - R_0\mathbf{k}^-)/k$.

Note 2. The derivations above are no more than a conspectus of the derivations provided in [5, pp. 74–80].

Note 3. Beckmann & Spizzichino [6] use the notation $\mathbf{v} \equiv \mathbf{k}^-$ and $\mathbf{p} \equiv \mathbf{k}^+$, and their formula (18a) on p. 22 is equivalent to (15).

Note 4. If the surface elevation $h(x_0, y_0) \equiv 0$ for all x_0 and y_0, then (18) takes the form

$$\psi_0^{\mathrm{sc}}(\mathbf{r}) = -\frac{ike^{ikr}}{4\pi r}\, c \int_{S_M} e^{ikAx_0 + ikBy_0} dx_0 dy_0 \,, \tag{25}$$

and if

$$S_M = \{(x_0, y_0) \mid -X \le x \le X,\ -Y \le y \le Y\}\,, \tag{26}$$

and $A_M \equiv 4XY$, then

$$\psi_0^{\mathrm{sc}}(\mathbf{r}) = \psi_0^{\mathrm{sc}}(r, \theta_1, \theta_2, \theta_3) = -\frac{ike^{ikr}}{4\pi r}\, cA_M \operatorname{sinc}(kAX) \operatorname{sinc}(kAY)\,, \tag{27}$$

where $\operatorname{sinc}(t) \equiv \sin(t)/t$. Equation (27) describes scattering of a plane wave from a flat $2X$-by-$2Y$ rectangular plate.

Note 5. Beckmann & Spizzichino [6] use the normalised amplitude which they define as

$$\rho(\mathbf{r}) \equiv \frac{\psi^{\mathrm{sc}}(\mathbf{r})}{\widehat{\psi}_0^{\mathrm{sc}}(r, \theta_1)}\,, \tag{28}$$

where

$$\widehat{\psi}_0^{\mathrm{sc}}(r, \theta_1) \equiv \psi_0^{\mathrm{sc}}(r, \theta_1, \theta_1, 0)|_{R_0=-1} = \frac{ike^{ikr}}{2\pi r} A_M \cos\theta_1\,. \tag{29}$$

A correction factor 1/2 is required to make the equation (1) on p. 26 of [6] to be equivalent to the result (18).

Note 6. It is noted in [5, p. 76] that instead of equation (2) one can use the equation

$$\psi^{\mathrm{sc}}(\mathbf{r}) = \int_{S_0} \left[\frac{\partial \psi^{\mathrm{sc}}(\mathbf{r}_0)}{\partial n_0} G(\mathbf{r} - \mathbf{r}_0) - \psi^{\mathrm{sc}}(\mathbf{r}_0) \frac{\partial G(\mathbf{r} - \mathbf{r}_0)}{\partial n_0} \right] dS_0 \tag{30}$$

as a starting point. This will give

$$\psi^{\mathrm{sc}}(\mathbf{r}) = \frac{ie^{ikr}}{4\pi r} \int_{S_0} R_0(\mathbf{k}^- \cdot \mathbf{n}_0) e^{i\mathbf{k}^- \cdot \mathbf{r}_0} dS_0$$

instead of (12). Equation (18) will have the same form, but for the coefficients a, b and c we will have to use the components of the vector $(-R_0\mathbf{k}^-/k)$.

3 Coherent and diffuse fields

3.1 Coherent and diffuse fields and their intensities

The coherent and diffuse fields are defined as follows

$$\text{Coherent Field} \equiv \langle \psi^{\mathrm{sc}} \rangle,$$

$$\text{Diffuse Field} \equiv \psi^{\rm sc} - \langle \psi^{\rm sc} \rangle.$$

Throughout, $\langle \cdot \rangle$ is the mean (expectation) taken over all permissible realisations of the surface S_0. The intensities $I_{\rm c}$ and $I_{\rm d}$ of the coherent and diffuse fields are

$$I_{\rm c} \equiv |\langle \psi^{\rm sc} \rangle|^2 \qquad \text{and} \qquad I_{\rm d} \equiv |\psi^{\rm sc} - \langle \psi^{\rm sc} \rangle|^2 \,.$$

Using these definitions we obtain the relation that connects $\langle I \rangle$ [=the mean intensity of the total field], $I_{\rm c}$ [=the intensity of the coherent field], and $\langle I_{\rm d} \rangle$ [=the mean intensity of the diffuse field]

$$\langle I_{\rm d} \rangle \equiv \langle |\psi^{\rm sc}|^2 \rangle - |\langle \psi^{\rm sc} \rangle|^2 = \langle I \rangle - I_{\rm c} \,. \tag{31}$$

Once formula (18) has been obtained, both [5] and [6] proceed to analysis of the composition of the scattered field. They consider S_M of the form (26) and carry out integration by parts in x_0 and y_0 in the terms containing the derivatives $\partial h/\partial x_0$ and $\partial h/\partial y_0$ respectively, as a result of which $\psi^{\rm sc}(\mathbf{r})$ can be represented as

$$\psi^{\rm sc}(\mathbf{r}) = \psi_{-e}(\mathbf{r}) + \psi_e(\mathbf{r}) \,, \tag{32}$$

where

$$\psi_{-e}(\mathbf{r}) \equiv -\frac{ike^{ikr}}{4\pi r} 2F \int_{S_M} e^{ik\phi(x_0,y_0)} dx_0 dy_0 \tag{33}$$

$$F(\theta_1, \theta_2, \theta_3) \equiv \frac{1}{2}\left(\frac{Aa}{C} + \frac{Bb}{C} + c\right) \tag{34}$$

$$\phi(x_0, y_0) \equiv Ax_0 + By_0 + Ch(x_0, y_0) \tag{35}$$

and

$$\begin{aligned} \psi_e(\mathbf{r}) \equiv -\frac{ike^{ikr}}{4\pi r} \Bigg[& \frac{ia}{kC} \int_{-Y}^{Y} \left(e^{ik\phi(X,y_0)} - e^{ik\phi(-X,y_0)} \right) dy_0 \\ & + \frac{ib}{kC} \int_{-X}^{X} \left(e^{ik\phi(x_0,Y)} - e^{ik\phi(x_0,-Y)} \right) dx_0 \Bigg] \,. \end{aligned} \tag{36}$$

Note 7. Equation (34) can be written as

$$F = \frac{\mathbf{k}^- \cdot (\mathbf{k}^+ - R_0 \mathbf{k}^-)}{2Ck^2} = -\frac{R_0 |\mathbf{k}^-|^2}{2Ck^2} \,, \tag{37}$$

which shows that the term ψ_{-e} is invariant with respect to whether equation (2) or (30) is used as a starting point of derivations of (34). However, due to different definitions of a and b (see Note 6), the term ψ_e does depend on whether (2) or (30) are chosen for deriving (36). These aspects are discussed in more detail in [5, p. 83].

Beckmann & Spizzichino [6, p. 31] argue that the term $\psi_e(\mathbf{r})$ can be neglected. Ogilvy [5, p. 86] demonstrates that neglecting the term $\psi_e(\mathbf{r})$ would lead to an erroneous result when evaluating the coherent field. Comments in [6, p. 78] near equations (25) and (26) confirm that Beckmann & Spizzichino were aware of the necessity to account for the "edge effects" when evaluating the coherent field.

During evaluation of the mean intensity of the diffuse field both [6] and [5] assume that the edges of the surface $h(x_0, y_0)$ are fixed at the mean level, that is $h(\pm X, y_0) = 0$ and $h(x_0, \pm Y) = 0$. This assumption does allow us to neglect the terms associated with ψ_e, as we obtain

$$\langle I_\mathrm{d} \rangle = \langle |\psi_{-e}|^2 \rangle - |\langle \psi_{-e} \rangle|^2,$$

however it invalidates the result $\langle \psi^\mathrm{sc} \rangle = \langle e^{ikCh} \rangle \psi_0^\mathrm{sc}$ (see (4.34) [5, p. 86]) obtained for the coherent field.

In order to evade the complications associated with the edge term, we will adopt the Small Slope Approximation, which assumes that $\partial h/\partial x_0$ and $\partial h/\partial y_0$ are negligibly small. Use this approximation to rewrite (18) as

$$\psi^\mathrm{sc}(\mathbf{r}) = -\frac{ike^{ikr}}{4\pi r} c \int_{S_M} e^{ik[Ax_0 + By_0 + Ch(x_0,y_0)]} dx_0 dy_0 \,. \tag{38}$$

3.2 Assumptions about statistics of the scattering surface

Suppose that the random surface $z_0 = h(x_0, y_0)$ has the following two-point height probability distribution

$$p_2(h_1, h_2, \mathbf{R}) = \frac{\exp\left\{-\frac{h_1^2 - 2C_0(R)h_1h_2 + h_2^2}{2\sigma^2(1 - C_0^2(R))}\right\}}{2\pi\sigma^2\sqrt{1 - C_0^2(R)}} \,, \tag{39}$$

where $\mathbf{R}$ is the vector drawn from the first to the second point, R is the distance between them, h_1 and h_2 are random elevations at the first and the second point, that is

$$\mathbf{R} \equiv R_x \mathbf{e}_x + R_y \mathbf{e}_y,\ R \equiv \sqrt{R_x^2 + R_y^2},$$
$$h_1 \equiv h(x_0, y_0),\ h_2 \equiv h(x_0 - R_x, y_0 - R_y).$$

Following [5, 6] we consider $C_0(R)$ of the following form

$$C_0(R) = e^{-R^2/T^2} \,. \tag{40}$$

It can be readily verified that the height distribution of $z_0 = h(x_0, y_0)$ is Gaussian

$$p(h) = \int_{-\infty}^{\infty} p_2(h, h_2, \mathbf{R}) dh_2 = \frac{e^{-h^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \,, \tag{41}$$

and we also have: $\langle h \rangle = 0$ and $\langle h^2 \rangle = \sigma^2$.

It can also be shown that the autocorrelation coefficient of the random surface $z_0 = h(x_0, y_0)$ is $C_0(R)$, that is

$$\frac{\langle h(x_0, y_0) h(x_0 - R_x, y_0 - R_y) \rangle}{\langle h^2(x_0, y_0) \rangle} = C_0(R) \,. \tag{42}$$

Equations (40) and (42) allows us to interpret T as the correlation length.

The characteristic functions of $p(h)$ and $p_2(h_1, h_2, \mathbf{R})$ have the following form

$$\chi(s) \equiv \langle e^{ish} \rangle = \int_{-\infty}^{\infty} p(h) e^{ish} dh = \exp\left\{-s^2\sigma^2/2\right\} \tag{43}$$

$$\begin{aligned}\chi_2(s_1, s_2, R) \equiv \langle e^{is_1h_1+is_2h_2} \rangle &= \int_{\mathbf{R}^2} dh_1 dh_2 p_2(h_1, h_2, \mathbf{R}) e^{is_1h_1+is_2h_2} \\ &= \exp\left\{-\frac{\sigma^2}{2}(s_1^2 + 2C_0(R)s_1s_2 + s_2^2)\right\} \end{aligned} \tag{44}$$

Formula (44) acknowledges dependence of χ_2 on R by including this parameter in the list of its arguments.

3.3 Evaluation of intensities and their means

First, applying the expectation operator to (38), we obtain the coherent field:

$$\langle \psi^{\text{sc}}(\mathbf{r}) \rangle = -\frac{ike^{ikr}}{4\pi r} c \int_{S_M} e^{ik[Ax_0+By_0]} \chi(kC) dx_0 dy_0 = \psi_0^{\text{sc}}(\mathbf{r}) \chi(kC)\,, \tag{45}$$

where $\psi_0^{\text{sc}}(\mathbf{r})$ is defined by (25) and is the field resulted from scattering from a flat surface ($h \equiv 0$).

Note 8. In [5] equation (45) is obtained without resorting to the small slope approximation (see also equation (25), [6, p. 79]).

The intensity of the coherent field thus has the form

$$I_{\text{c}} \equiv |\langle \psi^{\text{sc}} \rangle|^2 = I_0 |\chi(kC)|^2 = I_0 e^{-g}\,, \tag{46}$$

where g is the roughness defined as

$$g \equiv k^2 C^2 \sigma^2 = k^2 \sigma^2 (\cos\theta_1 + \cos\theta_2)^2\,. \tag{47}$$

Next we evaluate the mean intensity of the diffuse field

$$\begin{aligned}\langle I_{\text{d}} \rangle &= \langle I \rangle - I_{\text{c}} \\ &= \frac{k^2c^2}{(4\pi r)^2} \int_{S_M \times S_M} e^{ik[A(x_0-x_1)+B(y_0-y_1)]} \\ &\qquad \times \left(\left\langle e^{ikC(h_0-h_1)} \right\rangle - \left\langle e^{ikCh_0} \right\rangle \left\langle e^{-ikCh_0} \right\rangle\right) dx_0 dy_0 dx_1 dy_1 \\ &= \frac{k^2c^2}{(4\pi r)^2} \int_{-X}^{X} dx_1 \int_{-Y}^{Y} dy_1 \int_0^{2\pi} d\theta \int_0^{R_{\max}(x_1,y_1,\theta)} dRR \\ &\qquad \times e^{ik[AR\cos\theta+BR\sin\theta]} \left[\chi_2(kC, -kC, R) - \chi(kC)\bar{\chi}(kC)\right] \qquad (48) \\ &= \frac{k^2c^2}{8\pi r^2} \int_{-X}^{X} dx_1 \int_{-Y}^{Y} dy_1 \int_0^{R_{\max}(x_1,y_1,\theta)} dRRJ_0(kR\sqrt{A^2+B^2}) \\ &\qquad \times e^{ik[AR\cos\theta+BR\sin\theta]} \left[\chi_2(kC, -kC, R) - \chi(kC)\bar{\chi}(kC)\right]\,, \qquad (49)\end{aligned}$$

where the transition from (48) to (49) is based on the formula

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(x\cos\theta + y\sin\theta)}d\theta = J_0(\sqrt{x^2+y^2})\,.$$

It follows from (40) that the function

$$\chi_2(kC,-kC,R) - \chi(kC)\bar{\chi}(kC) = e^{-k^2C^2\sigma^2[1-C_0(R)]} - e^{-k^2C^2\sigma^2}$$

rapidly decays as $R \to \infty$. If $T \ll \sqrt{X^2+Y^2}$, then we can simplify (48) by replacing the upper limit $R_{\max}(x_1, y_1, \theta)$ with ∞. This approximation removes dependence on x_1 and y_1 from the integrand, which allows us to carry out integration in those variables and obtain

$$\langle I_{\mathrm{d}}\rangle = \frac{k^2c^2}{8\pi r^2}A_M\int_0^\infty J_0(kR\sqrt{A^2+B^2})\left[e^{-g[1-C_0(R)]} - e^{-g}\right]RdR \tag{50}$$

Note 9. These derivations repeat, with minor modifications, the derivations provided in [5, p. 87] and [6, p. 78–79]. Ogilvy [5] does not use the small slope approximation. To turn formula (50) into the result obtained in [5] we simply have to replace c with $2F$ (compare (33) and (38), or see [5, p. 82]), which gives

$$\langle I_{\mathrm{d}}\rangle_{\mathrm{Ogilvy}} = \frac{k^2F^2}{2\pi r^2}A_M\int_0^\infty J_0(kR\sqrt{A^2+B^2})\left[e^{-g[1-C_0(R)]} - e^{-g}\right]RdR \tag{51}$$

The result analogous to (51) obtained in [6, p. 79] (equation (31)) has the form

$$D\{\rho\} = \frac{2\pi F_3^2}{A^2}\int_0^\infty J_0(v_{xy}\tau)[\chi_2(v_z,-v_z) - \chi(v_z)\chi^*(v_z)]\tau d\tau\,.$$

This has to be corrected first, instead of A^2 there should be just A. Next we recall that $\mathbf{v} \equiv \mathbf{k}^- = (v_x, v_y, v_z) = k(A, B, C)$, (see [6]: formula (2) p. 26), so $v_z = kC$ and

$$v_{xy} \equiv \sqrt{v_x^2+v_y^2} = k\sqrt{A^2+B^2}. \tag{52}$$

The coefficient F_3 is defined by equation (3) of [6, p. 29], where it has already been assumed that the reflection coefficient is 1. Comparison of equation (3) of [6, p. 29] with equation (4.30) of [5, p. 84] shows that

$$F_{\mathrm{BS}}^2 \equiv F_3^2 = F_{\mathrm{O}}^2/\cos^2\theta_1\,, \tag{53}$$

where $F_{\mathrm{O}} = F$ and F is given by (34) (or (37)) in which $R_0 = -1$, that is

$$F_{\mathrm{O}} \equiv F = \frac{1+\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos\theta_3}{\cos\theta_1+\cos\theta_2}\,. \tag{54}$$

If we divide (51) by $|\widehat{\psi}_0^{\,\mathrm{sc}}|^2$ (see (29)) and take the above considerations into account, we readily obtain that (51) and the result for $D\{\rho\}$ obtained in [6, p. 79] and reproduced above are equivalent.

Equations (50) and (51) become the same in the specular direction, for which we have $A = B = 0$, $F = c/2 = R_0 \cos\theta_1 = -\cos\theta_1$, $g = g_0 \equiv 4k^2\sigma^2\cos^2\theta_1$ and so

$$\langle I_{\mathrm{d}}\rangle = \langle I_{\mathrm{d}}\rangle_{\mathrm{spec}} = \frac{k^2\cos^2\theta_1}{2\pi r^2} A_M e^{-g_0} \int_0^\infty \left[e^{g_0 C_0(R)} - 1\right] R dR\,. \tag{55}$$

Using the expansion

$$\exp\left(g_0 e^{-R^2/T^2}\right) = \sum_{n=0}^{\infty} \frac{g_0^n e^{-nR^2/T^2}}{n!}$$

equation (55) can be transformed to

$$\begin{aligned}\langle I_{\mathrm{d}}\rangle_{\mathrm{spec}} &= \frac{k^2\cos^2\theta_1}{2\pi r^2} A_M e^{-g_0} \int_0^\infty \sum_{n=1}^{\infty} \frac{g_0^n e^{-nR^2/T^2}}{n!} R dR \\ &= \frac{T^2 k^2\cos^2\theta_1}{4\pi r^2} A_M e^{-g_0} \sum_{n=1}^{\infty} \frac{g_0^n}{n! n}\,,\end{aligned} \tag{56}$$

which gives the following expression for the mean square normalised amplitude

$$\langle|\rho|^2\rangle_{\mathrm{spec}} = \frac{|\widehat{\psi}_0^{\mathrm{sc}}(r,\theta_1)|^2 e^{-g_0} + \langle I_{\mathrm{d}}\rangle_{\mathrm{spec}}}{|\widehat{\psi}_0^{\mathrm{sc}}(r,\theta_1)|^2} = e^{-g_0} + \frac{T^2\pi}{A_M} e^{-g_0} \sum_{n=1}^{\infty} \frac{g_0^n}{n! n}\,. \tag{57}$$

Medwin [8] studied scattering in the specular direction and equation (57) has the same form as his key equation (8) (see [8, p. 8]). Note that Medwin's derivation of (57) is incomplete and contains errors[1]. Equation (57) can also be readily obtained from equation (57) of [6, p. 88] if we observe that in the specular direction $g = g_0$, $v_{xy} = 0$, $F_{\mathrm{BS}} = 1$ and $\rho_0 \equiv \mathrm{sinc}(v_x X)\,\mathrm{sinc}(v_y Y) = 1$.

Note 10. The statements (4.56) of [5, p. 92] and (44) of [6, p. 87] are erroneous. They both interpret

$$\frac{k^2 F^2}{2\pi r^2} A_M \int_0^\infty J_0(kR\sqrt{A^2+B^2}) e^{-g[1-C_0(R)]} R dR \tag{58}$$

as $\langle I\rangle$ when $g \to \infty$. The integral in (58) is divergent, so (58) is meaningless and can be interpreted as neither $\langle I\rangle$, nor $\langle I_{\mathrm{d}}\rangle$. Similarly, the expression

$$\frac{k^2 F^2}{2\pi r^2} A_M \int_0^\infty J_0(kR\sqrt{A^2+B^2}) e^{-g} R dR \tag{59}$$

is equally meaningless and cannot be interpreted as I_{c}. Also note that, because the integrals in (58) and (59) are divergent, we cannot replace in (48) the upper integration limit $R_{\max}(x_1, y_1, \theta)$ with ∞ if $\langle I\rangle$ and I_{c} are considered separately.

4 Asymptotic behaviour of $\langle I_{\mathrm{d}}\rangle$

In this section we verify the asymptotic formula describing the behaviour of $\langle I_{\mathrm{d}}\rangle$ when the roughness parameters $g \equiv k^2C^2\sigma^2$ is large for "gently undulated" surfaces [6, p. 194], which satisfy $4\sigma^2/T^2 \ll 1$.

[1] The last equation on page 7 of [8] before formula (8) is stated on page 8 is incorrect.

The main idea of the proof is outlined in [6, p. 87] and consists in replacing the integral in (50) with

$$\int_0^\infty J_0(kR\sqrt{A^2+B^2})e^{-gR^2/T^2}RdR\,. \tag{60}$$

Beckmann & Spizzichino argue that the main contribution to the integral in (50) is due to integration in the neighbourhood of $R = 0$, where $C_0(R)$ can be approximated by $\approx 1-\frac{R^2}{T^2}$. Using this approximation and dropping and adding terms insignificant when $g \to \infty$ leads to the expression (60), which can be evaluated analytically. The resulting estimate for the normalised intensity is equation (47) of [6, p. 87] reproduced below

$$\langle\rho\rho^*\rangle = \frac{\pi F^2T^2}{Ag}\exp\left(-\frac{v_{xy}^2T^2}{4g}\right) \qquad \text{(eqn. (47), [6, p. 87])} \tag{$*$}$$

where we have to recall that v_{xy} is given by (52) and F must be interpreted as F_{BS} (see (53)). Also, A in $(*)$ must be interpreted as A_M, that is the area of S_M.

Note 11. If we had $k^2T^2/4g \ll 1$, then retaining the asymptotic formula in the form $(*)$ would be meaningless, as the principal term would be simply the coefficient before the exponential. In our case $k^2T^2/4g = T^2/(4\sigma^2C^2)$ isn't a small value, so $(*)$ is exponentially small unless A^2+B^2 is equal to or close to zero, which corresponds to the specular direction.

Consider the integral in (50)

$$S(g) \equiv \int_0^\infty J_0(v_{xy}\tau)\left[e^{-g(1-\exp(-\tau^2/T^2))} - e^{-g}\right]\tau d\tau \tag{61}$$

where we have switched from R and $k\sqrt{A^2+B^2}$ to more concise τ and v_{xy}.

To verify the estimate $(*)$ and obtain the asymptotic formula for $\langle I_{\mathrm{d}}\rangle$ we have to study the behaviour of the integral (61) at large g. This is done in Appendix C where the following result is verified (see equation (C14))

$$S(g) = \frac{T^2}{2g}\left[\exp\left(-\frac{v_{xy}^2T^2}{4g}\right) + O(g^{-1})\right] \tag{62}$$

as $g \to \infty$.

Equation (50) and estimate (62) give

$$\langle I_{\mathrm{d}}\rangle = \frac{k^2c^2T^2}{16\pi r^2 g}A_M\left[\exp\left(-\frac{v_{xy}^2T^2}{4g}\right) + O(g^{-1})\right] \tag{63}$$

Note 12. We have been using the small slope approximation. If this approximation had not been used, the principal term of the asymptotic formula for $\langle I_{\mathrm{d}}\rangle$ would have the form (see [5, p. 92], equations (4.57) and (4.58)):

$$\langle I_{\mathrm{d}}\rangle_{\mathrm{Ogilvy}} \simeq \frac{k^2F^2T^2}{4\pi r^2 g}A_M\exp\left(-\frac{v_{xy}^2T^2}{4g}\right)\,. \tag{64}$$

Equation (64) can be obtained from the principal term of (63) by simply replacing c with $2F$ (compare (33) and (38), or see [5, p. 82]).

Note 13. If we divide (64) by

$$|\widehat{\psi}_0^{\rm sc}|^2 = \frac{k^2 A_M^2}{(2\pi r)^2} \cos^2\theta_1 \qquad \text{(see (29))}$$

and use (53) we obtain equation (47) of [6, p. 87] (see (∗) on page 11).

Note 14. The mean power reflection coefficient is defined as $\langle|\psi^{\rm sc}|^2\rangle/\Psi_0^2$, where Ψ_0 is the amplitude of the incident wave [6, p. 89]. In our case $\Psi_0 = 1$, so $\langle I\rangle$ ($\simeq \langle I_{\rm d}\rangle$ when $g \gg 1$) can be also interpreted as the mean power reflection coefficient. Beckmann & Spizzichino (see [6, p. 89], equation (63)) provide the following result for the principal term of the mean power reflection coefficient:

$$\langle|R_1|^2\rangle = \frac{A}{\pi r^2}\frac{v^4}{v_z^4}\cot^2\beta_0 \exp\left(-\frac{\tan^2\beta}{\tan^2\beta_0}\right) \qquad \text{(eqn. (63), [6, p. 89])} \qquad (\dagger)$$

where the angles β_0 and β are such that

$$\tan\beta_0 = \frac{2\sigma}{T} \quad \text{and} \quad \tan\beta = \frac{v_{xy}}{v_z}\,. \tag{65}$$

If we correct (†) by dividing its right-hand side by 4 and use (65) together with the relations

$$F = -\frac{R_0|\mathbf{k}^-|^2}{2Ck^2} = -\frac{R_0 v^2}{2v_z k} = \frac{v^2}{2v_z k} \quad (\text{as } R_0 = -1)$$

(see (37)) and $g = \sigma^2C^2k^2 = \sigma^2 v_z^2$, we obtain that (†) and (64) are equivalent.

Note 15. Because we have

$$v_{xy}^2 + v_z^2 = v^2 \quad \Rightarrow \quad \frac{v^2}{v_z^2} = \frac{v_{xy}^2}{v_z^2} + 1 = \tan^2\beta + 1 = \frac{1}{\cos^2\beta}$$

we can use an alternative form of (64)

$$\langle I_{\rm d}\rangle_{\rm Ogilvy} = \frac{A_M}{4\pi r^2}\frac{\cot^2\beta_0}{\cos^4\beta}\exp\left(-\frac{\tan^2\beta}{\tan^2\beta_0}\right) \tag{66}$$

This would coincide with equation (7) of [6, p. 251] if we take into account that there is a missing factor 1/4.

Note 16. While β can be readily interpreted as the angle between the z-axis and the vector $\mathbf{k}^- \equiv \mathbf{k}_{\rm inc} - \mathbf{k}_{\rm sc}$, interpretation of β_0 is less straightforward. Parameter $\tan^2\beta_0$ is connected with the mean square slope, which is defined for a two-dimensional surface $z_0 = h(x_0, y_0)$ as the ensemble average of the square tangent of the angle between the normal $\mathbf{n}_0$ to the surface S_0 and the vector $\mathbf{e}_z$. It can be readily shown that the mean square slope can be written as $\langle h_{x_0}^2 + h_{y_0}^2\rangle$, where, as before, the subscripts denote the partial derivatives in respective variables and $\langle\cdot\rangle$ denotes the expectation operator.

To establish the connection between $\langle h_{x_0}^2 + h_{y_0}^2\rangle$ and $\tan\beta_0$ consider the spatial processes $h_{x_0}(x_0, y_0)$ and $h_{y_0}(x_0, y_0)$ derived from the random process $h(x_0, y_0)$.

Let $P_0(\mathbf{k})$, $P_1(\mathbf{k})$ and $P_2(\mathbf{k})$ be the power spectrums of the processes h, h_{x_0} and h_{y_0}, and let $K_0(\mathbf{R})$, $K_1(\mathbf{R})$ and $K_2(\mathbf{R})$ be their autocorrelations. By (40) and (42), $K_0(\mathbf{R})$ is $\sigma^2\exp(-R^2/T^2)$. From the Wiener-Khintchine relations, $P_0(\mathbf{k})$ has the following form

$$P_0(\mathbf{k}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dR_x \int_{-\infty}^{\infty} dR_y K_0(\mathbf{R}) e^{i\mathbf{k}\mathbf{R}} = \frac{\sigma^2 T^2}{4\pi} e^{-T^2k^2/4}\,.$$

Using the properties of the Fourier transform and the definition of the power spectrum, we obtain $P_1(\mathbf{k}) = k_x^2 P_0(\mathbf{k})$ and $P_2(\mathbf{k}) = k_y^2 P_0(\mathbf{k})$. By the Wiener-Khintchine relations we have

$$K_\ell(\mathbf{R}) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y P_\ell(\mathbf{k}) e^{i\mathbf{kR}} \qquad (\ell = 1,\, 2)\,.$$

Hence

$$K_1(\mathbf{R}) = -\frac{\partial^2}{\partial R_x^2} K_0(\mathbf{R}) \quad \text{and} \quad K_2(\mathbf{R}) = -\frac{\partial^2}{\partial R_y^2} K_0(\mathbf{R})$$

and so

$$\begin{aligned} \langle h_{x_0}^2 \rangle &\equiv K_1(\mathbf{0}) = -\left.\frac{\partial^2}{\partial R_x^2} K_0(\mathbf{R})\right|_{\mathbf{R}=\mathbf{0}} = \frac{2\sigma^2}{T^2} \\ \langle h_{y_0}^2 \rangle &\equiv K_2(\mathbf{0}) = -\left.\frac{\partial^2}{\partial R_y^2} K_0(\mathbf{R})\right|_{\mathbf{R}=\mathbf{0}} = \frac{2\sigma^2}{T^2}\,. \end{aligned}$$

This gives us the equation connecting the mean square slope of the scattering surface and $\tan^2 \beta_0$

$$\langle h_{x_0}^2 + h_{y_0}^2 \rangle = \frac{4\sigma^2}{T^2} = \tan^2 \beta_0.$$

Using optical measurements Cox & Munk [9] obtain regression formulae expressing $\langle h_{x_0}^2 \rangle$, $\langle h_{y_0}^2 \rangle$ and $\langle h_{x_0}^2 + h_{y_0}^2 \rangle$ in terms of the wind speed. This was used by Medwin [8] to generate correlation lengths T for the wind speeds of interest. Note that Medwin [8] appears to erroneously apply the Cox & Munk [9] regression formula for $\langle h_{x_0}^2 + h_{y_0}^2 \rangle$ to $\langle h_{x_0}^2 \rangle = 2\sigma^2/T^2$.

5 Energy flux calculations

In this section we examine the rates of outgoing energies of the diffuse and coherent components of the scattered field. First, in Section 5.1, we provide calculations for scattering by a flat horizontal plate. These calculations show that all energy incident on a perfectly reflecting flat plate is entirely reflected in a narrow beam directed along the specular axis. Next, in Section 5.2, we consider a patch of a general gently undulated surface and evaluate the total flux Π_d of the diffuse energy of the scattered field for the situation when the roughness parameter is large. We show that if the incident wave vector $\mathbf{k}_\mathrm{inc}$ is separated from the horizontal direction then the total diffuse flow Π_d is equal to the incident energy flow Π_0.

For wave vectors $\mathbf{k}_\mathrm{inc}$ closely aligned with the horizontal direction, the expression for Π_d is more complicated but provides us with an explanation as to why the term V in the high-frequency surface loss formula has the form (96). The procedure described in [2, p. 54–56] uses the total diffuse flow in order to obtain the coherent reflection coefficient. If we strictly follow this procedure we obtain the required expression for SL_1. However, we believe that equation (3B-2) of [2, p. 54], which, in the notation of [2], has the following form

$$R + \int_0^{2\pi} \int_0^{\pi/2} \sigma \cos\phi_\mathrm{r} d\phi_\mathrm{r} d\theta_\mathrm{r} = 1\,, \qquad \text{(3B-2, [2, p. 54])}$$

is valid only for a vertically incident wave. For arbitrary grazing angles ϕ_i equation (3B-2, [2, p. 54]) should read

$$R\sin\phi_\mathrm{i} + \int_0^{2\pi}\int_0^{\pi/2} \sigma\cos\phi_\mathrm{r} d\phi_\mathrm{r} d\theta_\mathrm{r} = \sin\phi_\mathrm{i}\,.$$

The new expressions for SL_1 resulted from this correction are provided in Section 5.3 in Notes 18 and 19.

5.1 Energy flux calculation for reflection from a flat plate

The field due to reflection from a rectangular flat horizontal plate (26) is given by (26) (see Note 3)

$$\psi_0^{\mathrm{sc}}(\mathbf{r}) = -\frac{ike^{ikr}}{4\pi r}\, cA_M \,\mathrm{sinc}(kAX)\,\mathrm{sinc}(kAY)\,.$$

The total rate of energy reflected from the plate (26) is given by

$$\begin{aligned}\Pi_0 &\equiv \int_{-\pi}^{\pi} d\theta_3 \int_0^{\pi/2} d\theta_2 \sin\theta_2 r^2 |\psi_0^{\mathrm{sc}}|^2 \\ &= \int_{-\pi}^{\pi} d\theta_3 \int_0^{\pi/2} d\theta_2 \sin\theta_2 \frac{c^2A_M^2}{16XY}\left(\frac{\sin^2(kXA)}{\pi kXA^2}\right)\left(\frac{\sin^2(kYB)}{\pi kYB^2}\right)\,. \end{aligned} \tag{67}$$

To evaluate the integral (67) note that both kX and kY are large and that the sequence

$$S_m(x) \equiv \frac{\sin^2(mt)}{\pi m t^2} \tag{68}$$

is a δ-sequence as $m\to\infty$, that is

$$\lim_{m\to\infty}\int_{-\infty}^{\infty} S_m(t) f(t)dt = f(0) \tag{69}$$

for any smooth $f(t)$ vanishing outside a finite segment. The result (69) can be readily verified if we make use of the following facts:

$$\lim_{m\to\infty} S_m(t) = \begin{cases}\infty, & \text{if } t=0 \quad (\text{as } S_m(0)=m/\pi)\\ 0, & \text{if } t\neq 0\end{cases} \tag{70}$$

and

$$\int_{-\infty}^{\infty}\frac{\sin^2 mt}{\pi m t^2}dt = 1\,. \tag{71}$$

By (70), the integrand of (67) vanishes outside a small neighbourhood of the point where $A^2+B^2=0$ (that is, where we have $\theta_2=\theta_1$ and $\theta_3=0$), which corresponds to the specular direction[2]. Therefore the integration domain can be reduced to a small cone of directions which surrounds the specular axis. In order to make use of (69) we change from θ_2 and θ_3 to new integration variables

$$\xi = A(\theta_2,\theta_3), \qquad \eta = B(\theta_2,\theta_3)\,, \tag{72}$$

[2]This fact and equation (46) explain why the coherent field is described as a *specular spike* [10]

where A and B are given by (19) and (20).

The Jacobian of the transformation defined by (72) is

$$J(\theta_2, \theta_3) \equiv \frac{\partial(\xi, \eta)}{\partial(\theta_2, \theta_3)} = \begin{bmatrix} A_{\theta_2} & A_{\theta_3} \\ B_{\theta_2} & B_{\theta_3} \end{bmatrix} = \begin{bmatrix} -\cos\theta_2 \cos\theta_3 & \sin\theta_2 \sin\theta_3 \\ -\cos\theta_2 \sin\theta_3 & -\sin\theta_2 \cos\theta_3 \end{bmatrix}$$

and we have

$$\det J = \cos\theta_2 \sin\theta_2 \quad \Rightarrow \quad \frac{1}{|\det J|} = \frac{1}{\cos\theta_2 \sin\theta_2} .$$

Equation (67) transforms to

$$\Pi_0 \simeq \int d\xi \int d\eta \, \frac{c^2 A_M}{4\cos\theta_2}\bigg|_{\substack{\theta_2=\theta_2(\xi,\eta) \\ \theta_3=\theta_3(\xi,\eta)}} \left(\frac{\sin^2(kX\xi)}{\pi kX\xi^2}\right) \left(\frac{\sin^2(kY\eta)}{\pi kY\eta^2}\right) \tag{73}$$

where integration is carried out in small neighbourhoods of $\xi = 0$ and $\eta = 0$. The pair $\theta_2 = \theta_2(\xi, \eta)$ and $\theta_3 = \theta_3(\xi, \eta)$ used in (73) is a transformation inverse to (72).

Now use (70) to obtain

$$\Pi_0 \simeq \frac{c^2 A_M}{4\cos\theta_2}\bigg|_{\substack{\theta_2=\theta_2(0,0) \\ \theta_3=\theta_3(0,0)}} = A_M \cos\theta_1 , \tag{74}$$

where we have used (24) (with the substitution $R_0 = -1$) and the fact that $\theta_2(0, 0) = \theta_1$ and $\theta_3(0, 0) = 0$.

The incident field $e^{i\mathbf{k}_{\mathrm{inc}}\cdot\mathbf{r}}$ results in acoustic energy flowing in the direction $\mathbf{k}_{\mathrm{inc}}$. The rate at which this flow delivers the energy to the horizontal plate of area A_M is $A_M \cos\theta_1$. The result (74) is intuitively satisfying as it shows that under the Kirchhoff approximation, upon which the solution ψ_0^{sc} is based, all energy delivered to a flat perfectly reflecting plate is entirely reflected in a narrow beam of directions centred at the specular axis.

5.2 Calculation of the total diffuse energy flux

The rate at which the surface S_0 scatters the incident field $e^{i\mathbf{k}_{\mathrm{inc}}\cdot\mathbf{r}}$ into the diffuse energy is given by the equation

$$\Pi_{\mathrm{d}} = \int_{-\pi}^{\pi} d\theta_3 \int_0^{\pi/2} d\theta_2 \sin\theta_2 r^2 \langle I_{\mathrm{d}} \rangle \tag{75}$$

where $\langle I_{\mathrm{d}} \rangle$ is given by (50). If the roughness parameter g is large, equation (63) (or (64), if we do not use the small slope approximation) is valid and (75) takes the form

$$\Pi_{\mathrm{d}} = A_M \cot^2\beta_0 \int_{-\pi}^{\pi} d\theta_3 \int_0^{\pi/2} d\theta_2 Q(\theta_2, \theta_3) \exp\left(-\frac{\tan^2\beta}{\tan^2\beta_0}\right) , \tag{76}$$

where $Q(\theta_2, \theta_3)$ is given by

$$Q(\theta_2, \theta_3) = \frac{k^2 c^2 T^2 \sin\theta_2}{16\pi g} \frac{4\sigma^2}{T^2} = \frac{4\cos^2\theta_1 \sin\theta_2}{4\pi(\cos\theta_1 + \cos\theta_2)^2} = \frac{\cos^2\theta_1 \sin\theta_2}{\pi(\cos\theta_1 + \cos\theta_2)^2} , \tag{77}$$

in the small slope approximation, and

$$Q(\theta_2,\theta_3) = \frac{k^2F^2T^2\sin\theta_2}{4\pi g}\frac{4\sigma^2}{T^2} = \frac{F^2\sin\theta_2}{\pi(\cos\theta_1+\cos\theta_2)^2}$$
$$= \frac{(1+\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\cos\theta_3)^2\sin\theta_2}{\pi(\cos\theta_1+\cos\theta_2)^4}\,, \tag{78}$$

if we use (64) obtained without assuming the small slope approximation.

The integral (76) can be estimated at large values of

$$\mu \equiv \cot^2\beta_0 \equiv \frac{T^2}{4\sigma^2} \tag{79}$$

using a multi-dimensional form of the Laplace method [11, Ch. 2]. Let

$$\Phi(\theta_2,\theta_3) \equiv -\tan^2\beta\,. \tag{80}$$

The maximum of value of $\Phi(\theta_2,\theta_3)$ on $\Omega \equiv \{(\theta_2,\theta_3)\in[-\pi,\pi]\times[0,\pi/2]\}$ is zero and it is achieved if and only if both A and B are zeros (see (65) and (19)–(21)), which happens at the point $(\theta_2,\theta_3)=(\theta_1,0)$ corresponding to the specular direction.

If θ_1 is separated from $\pi/2$, then $(\theta_1,0)$ is an internal point of the integration domain and application of the Laplace method (e.g. [11, Theorem 4.1, p. 74]) gives

$$\Pi_\mathrm{d} \simeq \mu A_M \exp\Big\{\mu\Phi(\theta_1,0)\Big\}\left(\frac{2\pi}{\mu}\right)\frac{Q(\theta_1,0)}{\sqrt{|\det H(\theta_1,0)|}}\,, \tag{81}$$

where μ and Φ are given by (79) and (80), and $H(\theta_2,\theta_3)$ is the Hessian matrix of $\Phi(\theta_3,\theta_3)$ defined as

$$H(\theta_2,\theta_3) \equiv \begin{bmatrix}\Phi_{\theta_2\theta_2} & \Phi_{\theta_2\theta_3}\\ \Phi_{\theta_2\theta_3} & \Phi_{\theta_3\theta_3}\end{bmatrix}.$$

It can be shown that

$$H(\theta_1,0) = \begin{bmatrix}-\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\tan^2\theta_1\end{bmatrix} \quad\Rightarrow\quad \det H(\theta_1,0) = \frac{1}{4}\tan^2\theta_1\,. \tag{82}$$

Substitution of $\theta_2=\theta_1$ and $\theta_3=0$ into (77) and (78) gives

$$Q(\theta_1,0) = \frac{\sin\theta_1}{4\pi} \tag{83}$$

for both cases. Using (82), (83) and the fact that $\Phi(\theta_1,0)=0$ we obtain that (81) has the form

$$\Pi_\mathrm{d} \simeq \frac{2\pi Q(\theta_1,0)A_M}{\sqrt{|\det H(\theta_1,0)|}} = A_M\cos\theta_1\,. \tag{84}$$

If θ_1 is allowed to approach $\pi/2$ we can use the following argument. Since in (76) $\tan^{-2}\beta_0$ is large, the main contribution to the integral is made by permissible directions which belong to a small cone $\mathcal{K}_\delta$ around the specular axis $(\theta_2,\theta_3)=(\theta_1,0)$ where $\tan^2\beta=0$

$$\mathcal{K}_\delta \equiv \{(\theta_2,\theta_3)\in\Omega \mid \theta_1-\delta\le\theta_2\le\theta_1+\delta,\ -\delta\le\theta_3\le\delta\}$$

Because $(\theta_1, 0)$ is a stationary point of $\tan^2\beta$ and (cf. (82))

$$\frac{\partial^2}{\partial\theta_2^2}\left[\tan^2\beta\right]\Big|_{\substack{\theta_2=\theta_1\\ \theta_3=0}} = \frac{1}{2} \tag{85}$$

$$\frac{\partial^2}{\partial\theta_3^2}\left[\tan^2\beta\right]\Big|_{\substack{\theta_2=\theta_1\\ \theta_3=0}} = \frac{1}{2}\tan^2\theta_1 \tag{86}$$

$$\frac{\partial^2}{\partial\theta_2\theta_3}\left[\tan^2\beta\right]\Big|_{\substack{\theta_2=\theta_1\\ \theta_3=0}} = 0 \tag{87}$$

we can use Taylor's expansion and replace $\tan^2\beta$ inside $\mathcal{K}_\delta$ with

$$\tan^2\beta \simeq \frac{1}{4}\left[(\theta_2-\theta_1)^2 + \tan^2\theta_1\theta_3^2\right] \tag{88}$$

Approximating $Q(\theta_2, \theta_3)$ by its value at $(\theta_2, \theta_3) = (\theta_1, 0)$ (see (83)) and using (88) we transform (76) to the form

$$\Pi_\mathrm{d} \simeq \frac{A_M\sin\theta_1}{4\pi}\cot^2\beta_0\int_{-\delta}^{\delta}d\theta_3\int_{\theta_1-\delta}^{\min\left(\frac{\pi}{2},\theta_1+\delta\right)}d\theta_2 e^{-\frac{\cot^2\beta_0}{4}\left[(\theta_2-\theta_1)^2+\tan^2\theta_1\theta_3^2\right]}$$

The presence of the large parameter in the rapidly decaying exponential allows us to extend the limits of integration without introducing a considerable error

$$\Pi_\mathrm{d} \simeq \frac{A_M\sin\theta_1}{4\pi}\cot^2\beta_0\int_{-\infty}^{\infty}d\theta_3\int_{-\infty}^{\pi/2}d\theta_2 e^{-\frac{\cot^2\beta_0}{4}\left[(\theta_2-\theta_1)^2+\tan^2\theta_1\theta_3^2\right]} \tag{89}$$

Changing in (89) to new variables

$$\xi = \frac{1}{2}\cot(\beta_0)(\theta_2-\theta_1), \qquad \eta = \frac{1}{2}\cot(\beta_0)\tan(\theta_1)\theta_3$$

we obtain

$$\begin{aligned}
\Pi_\mathrm{d} &\simeq \frac{A_M\cos\theta_1}{\pi}\int_{-\infty}^{\infty}e^{-\eta^2}d\eta\int_{-\infty}^{\frac{1}{2}(\pi/2-\theta_1)\cot\beta_0}e^{-\xi^2}d\xi \\
&= \frac{A_M\cos\theta_1}{\sqrt{\pi}}\int_{-\frac{1}{2}(\pi/2-\theta_1)\cot\beta_0}^{\infty}e^{-\xi^2}d\xi \\
&= \frac{1}{2}A_M\cos\theta_1\,\mathrm{erfc}\left(-\frac{1}{2}\left(\frac{\pi}{2}-\theta_1\right)\cot\beta_0\right)
\end{aligned} \tag{90}$$

where $\mathrm{erfc}(w)$ is the complementary error function

$$\mathrm{erfc}(w) \equiv \frac{2}{\sqrt{\pi}}\int_w^\infty e^{-t^2}dt\,.$$

Equation (90) takes a more compact form if it is written in terms the grazing angle $\alpha_1 \equiv \pi/2-\theta_1$

$$\Pi_\mathrm{d} \simeq \frac{1}{2}A_M\sin\alpha_1\,\mathrm{erfc}\left(-\frac{\alpha_1\cot\beta_0}{2}\right)\,. \tag{91}$$

Note 17. Let us summarise the assumptions used in the derivation of (91). In addition to assumptions that guarantee applicability of the Kirchhoff theory (see [5, Section 4.2]) we have also assumed the following:

1. The two-point height distribution of the scattering surface is Gaussian (see (39)) and the autocorrelation coefficient has an exponential form (40).

2. The correlation distance T is much smaller than the size of the patch: $T \ll X$ and $T \ll Y$. This is required for stochastic treatment utilising the stationarity of the random surface [5, pp. 24-26] and this assumption was also used when (49) was transformed to (50).

3. The value of the roughness parameter is large: $g = 4k^2\sigma^2 \cos^2\theta_1 \gg 1$ (here $\theta_2 = \theta_1$).

4. We consider two-dimensional *gently undulated* surfaces, for which $\tan^2\beta_0 \equiv \frac{4\sigma^2}{T^2} \ll 1$. Note that by Note 16 this requirement is consistent with the adopted small slope approximation.

5. Shadowing effects are neglected.

Numerically the complementary error function is efficiently evaluated using the Chebyshev fitting (e.g. [12, pp. 220–221]), so the form (91) hardly required any further reduction. However, the following asymptotic formula[3] for erfc

$$\operatorname{erfc}(-u) = 2 - \frac{e^{-u^2}}{\sqrt{\pi}}\left(\frac{1}{u} + O(u^{-2})\right), \quad \text{as } u \to \infty, \tag{92}$$

has resulted in further transformation of (91) to

$$\Pi_\mathrm{d} \simeq A_M \sin\alpha_1 \left[1 - \frac{\exp\left(-\frac{\cot^2\beta_0\,\alpha_1^2}{4}\right)}{\sqrt{\pi}\,\alpha_1 \cot\beta_0}\right].$$

Formula (92) is valid when u is large; when $u \to 0$, the right-hand side of (92) tends to $-\infty$, whereas we must have

$$\operatorname{erfc}(-u) \geq \operatorname{erfc}(0) = 1\,. \tag{93}$$

The fact that (92) is singular at $u = 0$ and the requirement (93) have apparently resulted in the following formula (e.g. [1, 2])

$$\begin{aligned}\Pi_\mathrm{d} &\simeq A_M \sin\alpha_1 \max\left(\frac{\operatorname{erfc}(0)}{2},\left[1 - \frac{\exp\left(-\frac{\cot^2\beta_0\,\alpha_1^2}{4}\right)}{\sqrt{\pi}\,\alpha_1\cot\beta_0}\right]\right)\\ &\simeq A_M \max\left(\frac{\sin\alpha_1}{2},\ \sin\alpha_1\left[1 - \frac{\exp\left(-\frac{\cot^2\beta_0\,\alpha_1^2}{4}\right)}{\sqrt{\pi}\,\alpha_1\cot\beta_0}\right]\right).\end{aligned} \tag{94}$$

[3] Formula (92) can be readily obtained using integration by parts (e.g. [13, Ch. 3, Section 1]).

5.3 The acoustic surface loss formula

The high frequency acoustic surface loss formula used in propagation modelling (e.g. [1, p. 45], [2, p. 56], [3]) has the following form

$$\mathrm{SL}_1 = -10\log_{10}(1 - V) \tag{95}$$

where

$$V = \max\left(\frac{\sin\alpha_1}{2},\ \sin\alpha_1\left[1 - \frac{\exp\left(-\frac{\cot^2\beta_0\,\alpha_1^2}{4}\right)}{\sqrt{\pi}\,\alpha_1\cot\beta_0}\right]\right)\,. \tag{96}$$

The derivations of equation (94) clarify the nature and the composition (equation (96)) of the term V that appears in (95).

Note 18. Evaluation of the surface loss SL_1 using the definition (cf. [14, eqn. (9.6.1), p. 199] or [15, eqn. (4), p. 230])

$$\mathrm{SL}_1 \equiv -10\log_{10}\frac{\Pi_0 - \Pi_\mathrm{d}}{\Pi_0}\,, \tag{97}$$

where Π_0 and Π_d are given by (74) and (94), gives a result different from (95). Namely, we obtain

$$\mathrm{SL}_1 = -10\log_{10}(1 - V/\sin\alpha_1) \tag{98}$$

with V having the same form (96) as in (95).

Note 19. If in (97) we use for Π_d equation (91) instead of (94), we obtain

$$\mathrm{SL}_1 = -10\log_{10}\left[\frac{1}{2}\operatorname{erfc}\left(\frac{\alpha_1\cot\beta_0}{2}\right)\right]\,. \tag{99}$$

6 Concluding Remarks

The findings of Section 5 demonstrate that the currently used form of the surface loss term SL_1 is erroneous. Our analysis identifies at least two flaws in the derivations leading to equations (95) and (96). One of these flaws is an apparent fix (94) required as a result of the application of asymptotic expansion (92) outside its validity region. The second flaw is more fundamental and consists of an incorrect assumption that the energy flux balance equation for a wave of vertical incidence can be applied to an arbitrarily incident wave. Our corrected expression for SL_1 is given by equation (99).

Equation (99) follows from equation (97) which implies that the coherent energy flow can be evaluated from the total and diffuse energy flows. If we use an alternative definition

$$\mathrm{SL}_\mathrm{c} = -10\log_{10}\frac{\Pi_\mathrm{c}}{\Pi_0}$$

where

$$\Pi_\mathrm{c} \equiv \int_{-\pi}^{\pi} d\theta_3 \int_0^{\pi/2} d\theta_2 \sin\theta_2 r^2 I_\mathrm{c} \tag{100}$$

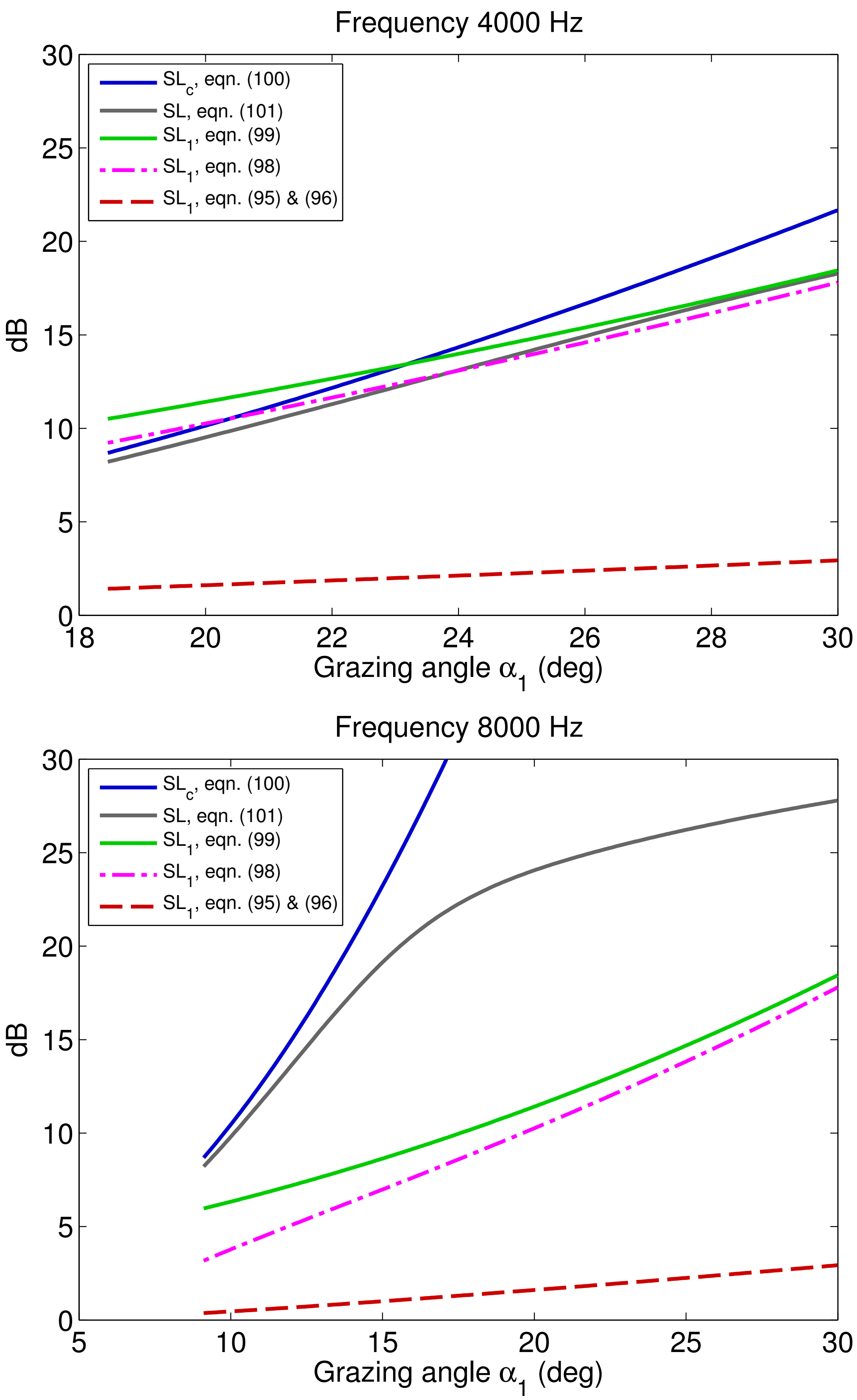

Figure 3: *Comparison of* $\mathrm{SL_c}$ *(eqn. (101)),* SL *(eqn. (102)), and the different forms of* $\mathrm{SL_1}$ *given by (95)–(96), (98), (99) and (101) for two wave frequencies 4 kHz (a) and 8 kHz (b).*

and I_c is the intensity of the coherent field, then equation (46) and an argument similar to that used in Section 5.1 give

$$\mathrm{SL_c} = 10 \log_{10} e^g \,. \tag{101}$$

This result is attributed to Eckart (see [16, eqn. (26), p. 569] and [3]).

Formula (63) shows that for gently undulated surfaces the diffuse field rapidly vanishes as we move from the specular direction. This is somewhat in contrast with the view (e.g. [5, p. 5–6]) that the specular field is coherent and the diffuse field has a wide angular spread. Medwin [8] used [6, eqn. (57), p. 88] to show that at small values of roughness parameter g the coherent component dominates in the specular direction, whereas at large values of g the field in the specular direction is dominated by the diffuse component. If the acoustic modelling uses incoherent addition of eigenray contributions, then the reflection coefficient considered in [8] (see equation [8, eqn. (8)] or our equation (57)) and the associated surface loss

$$\mathrm{SL} \equiv -10 \log_{10} \frac{\langle I \rangle|_{\theta_2=\theta_1,\,\theta_3=0}}{I_0} = -10 \log_{10} \langle |\rho|^2 \rangle_{\mathrm{spec}} \tag{102}$$

should be used instead of the coherent surface loss (101) or its alternatives, such as (98) or (99). Note that the area of the scattering patch is among the inputs of (57).

Figure 3 compares $\mathrm{SL_c}$ (eqn. (101)), SL (eqn. (102)), and the different forms of $\mathrm{SL_1}$ given by (95)–(96), (98), (99) and (101) for two wave frequencies 4 kHz and 8 kHz. The considered values of the sea state parameters (σ, T) are $(0.1333, 1.5766)$ meters, which corresponds to the wind speed of 5 m/sec. The considered dimensions of the patch are $X = Y = 10T \approx 15.58$ meters.

The results are plotted in the interval $\alpha_1 \in (\alpha_{\min}, \pi/6)$ where $\alpha_{\min}$ satisfies the condition $4k^2\sigma^2 \sin^2 \alpha_1 = g_{\min}$ and $g_{\min}$ is set at $g_{\min} = 2$ to keep the magnitude of the remainder in (C14) small (see Figure C1 (b)).

The plots of (101), (102) and (99) in Figure 3 (a) are reasonably close in the 4 kHz case, whereas they become quite separated in the 8 kHz case at larger grazing angles. Numerical integration of (75) is required in order to test whether or not the dissimilarity of plots in Figure 3 (b) is caused by the artefacts of the adopted asymptotic approximation. It also would be interesting to assess the difference between (67) and (74), and how this difference depends on the angle of incidence. Such tests are beyond the scope of the current work and can be pursued in a separate study.

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Appendix A List of Notation

This section provides the list of notation used in Sections 1–5.

$\mathbf{r}$	the radius-vector of the observer/receiver location
r	$r = \lvert\mathbf{r}\rvert$
$\mathbf{r}_0$	the radius-vector of the surface integration variable
ψ^{inc}	the complex amplitude of the incident pressure field
ψ^{sc}	the complex amplitude of the scattered pressure field
$\mathbf{k}_{\mathrm{inc}}$	the wave vector of the incident field
k	the wave number: $k \equiv \lvert\mathbf{k}_{\mathrm{inc}}\rvert = \lvert\mathbf{k}_{\mathrm{sc}}\rvert$
$\mathbf{k}_{\mathrm{sc}}$	the wave vector of the scattered field
$\mathbf{k}^{\pm}$	these vectors are defined as $\mathbf{k}^{\pm} = \mathbf{k}_{\mathrm{inc}} \pm \mathbf{k}_{\mathrm{sc}}$
S_0	the scattering surface, or a patch of the scattering surface
S_M	projection of S_0 on the mean plane
x_0, y_0 and z_0	Cartesian coordinates induced by the mean plane
$\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$	unit vectors of the Cartesian coordinate system induced by the mean plane
$\mathbf{n}_0$	a unit normal to S_0 at $\mathbf{r}_0$
$R_0(\mathbf{r}_0)$	the reflection coefficient (here mostly assumed to be -1)
$z_0 = h(x_0, y_0)$	a Cartesian representation of S_0
θ_1, θ_2, θ_3	angles describing directions of the incident and scattered fields, see Figure 2
A, B, C	components of vector $\mathbf{k}^-/k$, see equations (19)–(21)
a, b, c	components of vector $(\mathbf{k}^+ - R_0\mathbf{k}^-)/k$, see equations (22)–(24)
ψ_0^{sc}	the scattered field when S_0 is a flat horizontal rectangular plate, see equation (27)
$\mathrm{sinc}(s)$	the sinc function: $\mathrm{sinc}(s) \equiv \sin(s)/s$
X, Y	size parameters for rectangular S_M, see (26)
A_M	the area of S_M in the rectangular case: $A_M = 4XY$
$\widehat{\psi}_0^{\mathrm{sc}}$	ψ_0^{sc} considered at specular scattering direction and $R_0 = -1$, see equation (29)
I_{c}	the intensity of the coherent component
I_{d} and $\langle I_{\mathrm{d}}\rangle$	the intensity and the mean intensity of the diffuse component
I and $\langle I\rangle$	the total intensity and its mean
ψ_e^{sc} and ψ_{-e}^{sc}	the edge term and the scattered field from which the edge term has been removed, see (36) and (33)
$F(\theta_1, \theta_2, \theta_3)$	notation for $\frac{1}{2}\left(\frac{A}{a} + \frac{B}{b} + c\right)$, see equations (34), (37) and (54)
$p(h)$	the probability density function of the surface height distribution

$p_2(h_1, h_2, \mathbf{R})$	the probability density function describing the two-point height distribution of S_0
$\langle\cdot\rangle$	the mean (expectation) taken over all permissible realisations of the surface S_0
$\mathbf{R}$	vector $\mathbf{R} = (R_x, R_y)$ describes the displacement of the second point in the 2-point height pdf with respect to the first point
R	$R = \sqrt{R_x^2 + R_y^2}$
$C_0(R)$	the autocorrelation coefficient, see equation (40)
T	the correlation length
$\chi(s)$ and $\chi_2(s_1, s_2)$	the characteristic functions associated with the distributions $p(h)$ and $p_2(h_1, h_2, \mathbf{R})$, see equations (43) and (44)
$\chi_2(s_1, s_2, R)$	same as $\chi_2(s_1, s_2)$
$K_\ell(\mathbf{R})$ and $P_\ell(\mathbf{k})$	autocorrelations and power spectrums of the spatial processes $h(x_0, y_0)$ $(\ell = 0)$, $h_{x_0}(x_0, y_0)$ $(\ell = 1)$ and $h_{y_0}(x_0, y_0)$ $(\ell = 2)$
I_0	$I_0 \equiv \lvert\psi_0^{\mathrm{sc}}(\mathbf{r})\rvert^2$
σ	the standard deviation of surface heights
g	the roughness parameter $g = k^2\sigma^2(\cos\theta_1 + \cos\theta_2)$, see equation (47)
g_0	the roughness parameter for the specular direction, that is $g_0 = 4k^2\sigma^2\cos^2\theta_1$
$J_0(u)$	the Bessel function of order 0
$\mathbf{v}$	same as $\mathbf{k}^-$, $\mathbf{v} = (v_x, v_y, v_z) = k(A, B, C)$
v_{xy}	$v_{xy} = k\sqrt{A^2 + B^2}$, see (52)
$S(g)$	notation for $\int_0^\infty J_0(v_{xy}\tau)\left[e^{-g(1-\exp(-\tau^2/T^2))} - e^{-g}\right]\tau d\tau$, an integral factor in a reduced form of $\langle I_{\mathrm{d}}\rangle$, see equations (50) and (61)
$w(g)$	the coefficient in the $O(g^{-1})$ remainder of the large roughness approximation of $S(g)$, see equation (C15)
β_0 and β	angles associated with the mean square wave slope and the slope of vector $\mathbf{k}^-$, see equation (65) and Note 16
Π_0	the total rate of energy reflected from a horizontal flat rectangular plate, see (67)
$S_m(t)$	notation for $\frac{m}{\pi}\operatorname{sinc}^2(mt)$, see equation (68)
ξ and η	auxiliary variables, see equation (72)
Π_{d}	diffuse energy flux, see (75)
Π_{c}	coherent energy flux, see (100)
$Q(\theta_2, \theta_3)$	auxiliary function, see equations (77) and (78)
μ	notation for $\cot^2\beta_0$, see equation (79)
$\Phi(\theta_2, \theta_3)$	notation for $(-\tan^2\beta)$, see equation (80)
$H(\theta_2, \theta_3)$	the Hessian matrix of $\Phi(\theta_2, \theta_3)$
α_1	$\alpha_1 \equiv \pi/2 - \theta_1$

Appendix B The Helmholtz Integral

The Kirchhoff approximation is based on representation of the reflected field in terms of the Helmholtz integral, the derivation of which for two different types of the incident field is provided below. Let $U_{\rm r}(\mathbf{P})$ be the value of a complex amplitude of a pressure field at point $\mathbf{P}$. Here the subscript r is used to indicate that $U_{\rm r}(\mathbf{P})$ will be used to describe the reflected component of the total field. The function $U_{\rm r}(\mathbf{P})$ satisfies

$$\nabla^2_{\mathbf{P}} U_{\rm r} + k^2 U_{\rm r} = 0\,, \tag{B1}$$

$$U_{\rm r} = O(r^{-1}) \quad \text{and} \quad \frac{\partial U_{\rm r}}{\partial r} - ikU_{\rm r} = o(r^{-1})\,. \tag{B2}$$

where r is the distance from the origin. Equations (B2) are called the Sommerfeld radiation condition [7, eqn. (9A-12), p. 355].

The function

$$G(\mathbf{P}-\mathbf{Q}) \equiv -\frac{e^{ik|\mathbf{P}-\mathbf{Q}|}}{4\pi|\mathbf{P}-\mathbf{Q}|}\,, \tag{B3}$$

satisfies the equation

$$\nabla^2_{\mathbf{P}} G(\mathbf{P}-\mathbf{Q}) + k^2 G(\mathbf{P}-\mathbf{Q}) = \delta(\mathbf{P}-\mathbf{Q}) \tag{B4}$$

and the Sommerfeld radiation condition.

Let S be the boundary of a compact volume D and consider $U_{\rm r}(\mathbf{M})$ that satisfies (B1) outside D and has the asymptotic behaviour (B2) as $r \to \infty$. Let $\mathbf{M}$ be a point outside D, and let $\mathcal{D}_{\varepsilon,R}$ be a set comprising of a ball of radius R centred at $\mathbf{M}$, from which we have excluded the volume D and an ε-neighbourhood of the point $\mathbf{M}$. Also introduce the notation:

$B_r(\mathbf{M})$, the ball of radius r with the centre at $\mathbf{M}$,
$S_r(\mathbf{M})$, the sphere of radius r with the centre at $\mathbf{M}$.

The definition of $\mathcal{D}_{\varepsilon,R}$ can be written as $\mathcal{D}_{\varepsilon,R} \equiv B_R(\mathbf{M}) \setminus (D \cup B_\varepsilon(\mathbf{M}))$.

Consider

$$0 = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{\mathcal{D}_{\varepsilon,R}} U_{\rm r}(\mathbf{P}) \overbrace{\left[\nabla^2_{\mathbf{P}} G(\mathbf{P}-\mathbf{M}) + k^2 G(\mathbf{P}-\mathbf{M})\right]}^{\text{by (B4), this is zero}} dV_{\mathbf{P}}\,. \tag{B5}$$

Use the Gauss (divergence) theorem to transform the part of the integral associated with the first term in the square brackets.

$$\begin{aligned}
&\int_{\mathcal{D}_{\varepsilon,R}} U_{\rm r}(\mathbf{P})\nabla^2_{\mathbf{P}} G(\mathbf{P}-\mathbf{M})dV_{\mathbf{P}}\\
&\quad= \int_{\mathcal{D}_{\varepsilon,R}} \left[\nabla_{\mathbf{P}}\Big(U_{\rm r}(\mathbf{P})\nabla_{\mathbf{P}} G(\mathbf{P}-\mathbf{M})\Big) - \nabla_{\mathbf{P}} U_{\rm r}(\mathbf{P})\cdot\nabla_{\mathbf{P}} G(\mathbf{P}-\mathbf{M})\right] dV_{\mathbf{P}}\\
&\quad= -\int_S U_{\rm r}(\mathbf{P})\frac{\partial}{\partial n_{\mathbf{P}}} G(\mathbf{P}-\mathbf{M})dS_{\mathbf{P}} + \int_{S_R(\mathbf{M})} U_{\rm r}(\mathbf{P})\frac{\partial}{\partial n_{\mathbf{P}}} G(\mathbf{P}-\mathbf{M})dS_{\mathbf{P}}\\
&\qquad - \int_{S_\varepsilon(\mathbf{M})} U_{\rm r}(\mathbf{P})\frac{\partial}{\partial n_{\mathbf{P}}} G(\mathbf{P}-\mathbf{M})dS_{\mathbf{P}} - \int_{\mathcal{D}_{\varepsilon,R}} \nabla_{\mathbf{P}} U_{\rm r}(\mathbf{P})\cdot\nabla_{\mathbf{P}} G(\mathbf{P}-\mathbf{M})dV_{\mathbf{P}}
\end{aligned} \tag{B6}$$

where $\frac{\partial}{\partial n_{\mathbf{P}}} \equiv \mathbf{n_P} \cdot \nabla_{\mathbf{P}}$ denotes a derivative along the outward normal $\mathbf{n_P}$ to the integration surface at point $\mathbf{P}$.

Note that

$$\begin{aligned}\lim_{\varepsilon\to 0}&\left\{-\int_{S_\varepsilon(\mathbf{M})} U_{\mathrm{r}}(\mathbf{P})\frac{\partial}{\partial n_{\mathbf{P}}}G(\mathbf{P}-\mathbf{M})\,dS_{\mathbf{P}}\right\}\\ &= \lim_{\varepsilon\to 0}\int_{S_\varepsilon(\mathbf{M})} U_{\mathrm{r}}(\mathbf{P})\frac{e^{ik\varepsilon}}{4\pi}\left[-\frac{1}{\varepsilon^2}+\frac{ik}{\varepsilon}\right]dS_{\mathbf{P}}\\ &= \lim_{\varepsilon\to 0}\left\{4\pi\varepsilon^2\frac{U_{\mathrm{r}}(\mathbf{M})}{4\pi}\left[-\frac{1}{\varepsilon^2}+\frac{ik}{\varepsilon}\right]\right\} = -U_{\mathrm{r}}(\mathbf{M})\,. \qquad \text{(B7)}\end{aligned}$$

Next step

$$\begin{aligned}-&\int_{\mathcal{D}_{\varepsilon,R}} \nabla_{\mathbf{P}}U_{\mathrm{r}}(\mathbf{P})\cdot\nabla_{\mathbf{P}}G(\mathbf{P}-\mathbf{M})dV_{\mathbf{P}}\\ &= -\int_{\mathcal{D}_{\varepsilon,R}}\Big[\nabla_{\mathbf{P}}\Big(\nabla_{\mathbf{P}}U_{\mathrm{r}}(\mathbf{P})G(\mathbf{P}-\mathbf{M})\Big)-\nabla^2_{\mathbf{P}}U_{\mathrm{r}}(\mathbf{P})G(\mathbf{P}-\mathbf{M})\Big]\,dV_{\mathbf{P}} \qquad \text{(B8)}\end{aligned}$$

Note 1-a. Since $\nabla^2_{\mathbf{P}}U_{\mathrm{r}}(\mathbf{P}) + k^2U_{\mathrm{r}}(\mathbf{P}) = 0$, the second term in the integrand of (B8) combined with the second term in the integrand of (B5) gives zero.

Use the divergence theorem to transform the first term in (B8).

$$\begin{aligned}-\int_{\mathcal{D}_{\varepsilon,R}}\nabla_{\mathbf{P}}\Big(\nabla_{\mathbf{P}}U_{\mathrm{r}}(\mathbf{P})G(\mathbf{P}-\mathbf{M})\Big)\,dV_{\mathbf{P}} &= \int_S \frac{\partial U_{\mathrm{r}}(\mathbf{P})}{\partial n_{\mathbf{P}}}G(\mathbf{P}-\mathbf{M})dS_{\mathbf{P}}\\ -\int_{S_R(\mathbf{M})}\frac{\partial U_{\mathrm{r}}(\mathbf{P})}{\partial n_{\mathbf{P}}}G(\mathbf{P}-\mathbf{M})\,dS_{\mathbf{P}} &+ \int_{S_\varepsilon(\mathbf{M})}\frac{\partial U_{\mathrm{r}}(\mathbf{P})}{\partial n_{\mathbf{P}}}G(\mathbf{P}-\mathbf{M})\,dS_{\mathbf{P}}\,. \qquad \text{(B9)}\end{aligned}$$

Since $U_{\mathrm{r}}(\mathbf{P})$ is regular at $\mathbf{M}$ and $|G(\mathbf{P}-\mathbf{M})| = 1/(4\pi\varepsilon)$ when $\mathbf{P}\in B_\varepsilon(\mathbf{M})$, we have

$$\lim_{\varepsilon\to 0}\int_{S_\varepsilon(\mathbf{M})}\frac{\partial U_{\mathrm{r}}(\mathbf{P})}{\partial n_{\mathbf{P}}}G(\mathbf{P}-\mathbf{M})\,dS_{\mathbf{P}} = 0\,. \qquad \text{(B10)}$$

After substituting (B9) into (B6), we obtain upon application of the Sommerfeld radiation condition that the integral over $S_R(\mathbf{M})$ vanishes as $R\to\infty$:

$$\begin{aligned}\lim_{R\to\infty}&\int_{S_R(\mathbf{M})}\left[U_{\mathrm{r}}(\mathbf{P})\frac{\partial G(\mathbf{P}-\mathbf{M})}{\partial n_{\mathbf{P}}}-\frac{\partial U_{\mathrm{r}}(\mathbf{P})}{\partial n_{\mathbf{P}}}G(\mathbf{P}-\mathbf{M})\right]dS_{\mathbf{P}}\\ &= \lim_{R\to\infty}\int_{S_R(\mathbf{M})}\left[-ikU_{\mathrm{r}}(\mathbf{P})G(\mathbf{P}-\mathbf{M})+ikU_{\mathrm{r}}(\mathbf{P})G(\mathbf{P}-\mathbf{M})+o(R^{-2})\right]dS_{\mathbf{P}}\\ &= \lim_{R\to\infty}\int_{S_R(\mathbf{M})}\left[o(R^{-2})\right]dS_{\mathbf{P}} = 0\,. \qquad \text{(B11)}\end{aligned}$$

Hence, it follows from (B6), (B7), (B8), Note 1-a, (B9), (B10) and (B11) that (B5) is equivalent to

$$0 = -U_{\mathrm{r}}(\mathbf{M}) + \int_S\left[-U_{\mathrm{r}}(\mathbf{P})\frac{\partial G(\mathbf{P}-\mathbf{M})}{\partial n_{\mathbf{P}}}+\frac{\partial U_{\mathrm{r}}(\mathbf{P})}{\partial n_{\mathbf{P}}}G(\mathbf{P}-\mathbf{M})\right]dS_{\mathbf{P}}$$

or

$$U_{\mathrm{r}}(\mathbf{M}) = \int_S\left[\frac{\partial U_{\mathrm{r}}(\mathbf{P})}{\partial n_{\mathbf{P}}}G(\mathbf{P}-\mathbf{M})-U_{\mathrm{r}}(\mathbf{P})\frac{\partial G(\mathbf{P}-\mathbf{M})}{\partial n_{\mathbf{P}}}\right]dS_{\mathbf{P}} \qquad \text{(B12)}$$

Note 2-a. These derivations are essentially based on the formula

$$\nabla^2_{\mathbf{P}} G U_{\mathrm{r}} = \nabla_{\mathbf{P}} \Big(\nabla_{\mathbf{P}} G U_{\mathrm{r}} - G \nabla_{\mathbf{P}} U_{\mathrm{r}} \Big) + G \nabla^2_{\mathbf{P}} U_{\mathrm{r}} \tag{B13}$$

Substitution of (B13) into (B5), taking account of (B1), and application of the divergence theorem to the remaining term gives

$$0 = \lim_{\substack{\varepsilon\to 0 \\ R\to\infty}} \left\{ \int_{S_R(\mathbf{M})} \eta[G, U_{\mathrm{r}}] dS_{\mathbf{P}} - \int_{S_\varepsilon(\mathbf{M})} \eta[G, U_{\mathrm{r}}] dS_{\mathbf{P}} - \int_S \eta[G, U_{\mathrm{r}}] dS_{\mathbf{P}} \right\} \tag{B14}$$

where

$$\eta[G, U_{\mathrm{r}}] \equiv \frac{\partial G(\mathbf{P}-\mathbf{M})}{\partial n_{\mathbf{P}}} U_{\mathrm{r}}(\mathbf{P}) - G(\mathbf{P}-\mathbf{M}) \frac{\partial U_{\mathrm{r}}(\mathbf{P})}{\partial n_{\mathbf{P}}} \,. \tag{B15}$$

We have shown the following

$$\lim_{R\to\infty} \int_{S_R(\mathbf{M})} \eta[G, U_{\mathrm{r}}] dS_{\mathbf{P}} = 0 \qquad \text{[by the Sommerfeld condition (B2)]} \tag{B16}$$

$$\lim_{\varepsilon\to 0} \int_{S_\varepsilon(\mathbf{M})} \eta[G, U_{\mathrm{r}}] dS_{\mathbf{P}} = U_{\mathrm{r}}(\mathbf{M}) \quad \text{[by (B7) and (B10)]} \tag{B17}$$

so (B14) can be written as

$$U_{\mathrm{r}}(\mathbf{M}) = -\int_S \eta[G, U_{\mathrm{r}}] dS_{\mathbf{P}} \tag{B18}$$

which coincides with (B12).

By analogy with (B5), consider

$$0 = \lim_{\substack{\varepsilon\to 0 \\ R\to\infty}} \int_{\mathcal{G}_{\varepsilon,R}} G(\mathbf{P}-\mathbf{Q}) \left[\nabla^2_{\mathbf{P}} G(\mathbf{P}-\mathbf{M}) + k^2 G(\mathbf{P}-\mathbf{M}) \right] dV_{\mathbf{P}}, \tag{B19}$$

where $\mathbf{Q} \neq \mathbf{M}$, both $\mathbf{Q}$ and $\mathbf{M}$ are assumed to be outside D, but inside $B_R(\mathbf{M})$, and $\mathcal{G}_{\varepsilon,R} \equiv B_R(\mathbf{M}) \setminus (D \cup B_\varepsilon(\mathbf{M}) \cup B_\varepsilon(\mathbf{Q}))$.

Similarly to $U_{\mathrm{r}}(\mathbf{P})$ in $\mathcal{D}_{\varepsilon,R}$, the function $G(\mathbf{P}-\mathbf{Q})$ satisfies (B1) in $\mathcal{G}_{\varepsilon,R}$ for any positive ε and R. It also satisfies the Sommerfeld radiation condition. Using an argument similar to that used for (B18) we reduce (B19) to

$$\begin{aligned} G(\mathbf{M}-\mathbf{Q}) = & -\int_S \eta\left[G(\mathbf{P}-\mathbf{M}), G(\mathbf{P}-\mathbf{Q})\right] dS_{\mathbf{P}} \\ & - \lim_{\varepsilon\to 0} \int_{S_\varepsilon(\mathbf{Q})} \eta\left[G(\mathbf{P}-\mathbf{M}), G(\mathbf{P}-\mathbf{Q})\right] dS_{\mathbf{P}} \,, \end{aligned} \tag{B20}$$

where η is defined by (B15). The function $G(\mathbf{P}-\mathbf{M})$ is regular at $\mathbf{P} = \mathbf{Q}$, whereas $G(\mathbf{P}-\mathbf{Q})$ is singular ($\sim |\mathbf{P}-\mathbf{Q}|^{-1}$ as $\mathbf{P} \to \mathbf{Q}$).

$$\begin{aligned} & \lim_{\varepsilon\to 0} \int_{S_\varepsilon(\mathbf{Q})} \eta\left[G(\mathbf{P}-\mathbf{M}), G(\mathbf{P}-\mathbf{Q})\right] dS_{\mathbf{P}} \\ & \quad = -\lim_{\varepsilon\to 0} \int_{S_\varepsilon(\mathbf{Q})} G(\mathbf{P}-\mathbf{M}) \frac{\partial G(\mathbf{P}-\mathbf{Q})}{\partial n_{\mathbf{P}}} dS_{\mathbf{P}} = -G(\mathbf{Q}-\mathbf{M}) \,. \end{aligned} \tag{B21}$$

Substitute (B21) into (B20) to obtain

$$\int_S \eta\,[G(\mathbf{P}-\mathbf{M}), G(\mathbf{P}-\mathbf{Q})]\,dS_{\mathbf{P}} = 0\,. \tag{B22}$$

Now define $U(\mathbf{P}) = G(\mathbf{P}-\mathbf{Q}) + U_{\mathrm{r}}(\mathbf{P})$ and use equations (B18) and (B22) to obtain

$$U_{\mathrm{r}}(\mathbf{M}) = -\int_S \eta\,[G(\mathbf{P}-\mathbf{M}), U(\mathbf{P})]\;dS_{\mathbf{P}}\,, \tag{B23}$$

or

$$U_{\mathrm{r}}(\mathbf{M}) = \int_S \left[\frac{\partial U(\mathbf{P})}{\partial n_{\mathbf{P}}} G(\mathbf{P}-\mathbf{M}) - U(\mathbf{P})\frac{\partial G(\mathbf{P}-\mathbf{M})}{\partial n_{\mathbf{P}}}\right] dS_{\mathbf{P}}\,. \tag{B24}$$

The function $G(\mathbf{P}-\mathbf{Q})$ can be interpreted as an incident wave $U_{\mathrm{i}}(\mathbf{P})$ from a point source at $\mathbf{Q}$ and, given that the total solution $U(\mathbf{P}) = U_{\mathrm{i}}(\mathbf{P}) + U_{\mathrm{r}}(\mathbf{P})$ satisfies the boundary conditions at S, the function $U_{\mathrm{r}}(\mathbf{P})$ in this interpretation becomes the amplitude of the reflected wave. The right-hand side of equation (B24) is referred to (e.g. [5]) as the Helmholtz integral.

Formula (B24) also holds true if we use a plane wave for $U_{\mathrm{i}}(\mathbf{P})$, that is

$$U_{\mathrm{i}}(\mathbf{P}) = e^{iks\cos\theta + i\varphi}\,, \tag{B25}$$

where s is the distance from the $\mathbf{M}$ to $\mathbf{P}$ and θ is the angle between the wave vector $\mathbf{k}$ and vector $\mathbf{r_{MP}} \equiv \mathbf{r_M} - \mathbf{r_P}$. For example, if $U_{\mathrm{i}}(\mathbf{P}) = e^{i\mathbf{k}\cdot\mathbf{r_P}}$, then

$$U_{\mathrm{i}}(\mathbf{P}) = e^{i\mathbf{k}\cdot(\mathbf{r_{MP}}+\mathbf{r_M})} = e^{iks\cos\theta + i(\mathbf{k}\cdot\mathbf{r_M})}\,.$$

The function (B25) doesn't have singular points and satisfies the equation (B1) everywhere in $\mathbf{R}^3$ but doesn't satisfy the Sommerfeld radiation condition.

As in the case $U_{\mathrm{i}}(\mathbf{P}) = G(\mathbf{P}-\mathbf{Q})$, we start with an equation similar to equation (B19).

$$0 = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{\mathcal{D}_{\varepsilon,R}} U_{\mathrm{i}}(\mathbf{P})\left[\nabla^2_{\mathbf{P}} G(\mathbf{P}-\mathbf{M}) + k^2 G(\mathbf{P}-\mathbf{M})\right] dV_{\mathbf{P}}, \tag{B26}$$

Since $U_{\mathrm{i}}(\mathbf{P})$ is regular everywhere, we use here the same volume of integration $\mathcal{D}_{\varepsilon,R}$ as in (B5).

Using the same argument as in the derivation of formula (B14) we obtain

$$0 = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left\{\int_{S_R(\mathbf{M})} \eta[G, U_{\mathrm{i}}]dS_{\mathbf{P}} - \int_{S_\varepsilon(\mathbf{M})} \eta[G, U_{\mathrm{i}}]dS_{\mathbf{P}} - \int_S \eta[G, U_{\mathrm{i}}]dS_{\mathbf{P}}\right\}, \tag{B27}$$

where we can show as before (see (B17)) that

$$\lim_{\varepsilon\to 0} \int_{S_\varepsilon(\mathbf{M})} \eta[G, U_{\mathrm{i}}]dS_{\mathbf{P}} = U_{\mathrm{i}}(\mathbf{M}) = e^{i\varphi}\,. \tag{B28}$$

Because $U_{\mathrm{i}}(\mathbf{P})$ doesn't satisfy the Sommerfeld radiation condition the integral over $S_R(\mathbf{M})$ doesn't vanish as $R \to \infty$. Let us show that

$$\lim_{R\to\infty} \int_{S_R(\mathbf{M})} \eta[G, U_{\mathrm{i}}]dS_{\mathbf{P}} = e^{i\varphi}\,. \tag{B29}$$

Since, on $S_R(\mathbf{M})$,

$$\begin{aligned}\eta[G, U_\mathrm{i}] &= \frac{\partial}{\partial s}\left(-\frac{e^{iks}}{4\pi s}\right)e^{iks\cos\theta+i\varphi} - \left(-\frac{e^{iks}}{4\pi s}\right)\frac{\partial}{\partial s}e^{iks\cos\theta+i\varphi}\Big|_{s=R} \\ &= -\frac{e^{iks\cos\theta+i\varphi+iks}}{4\pi}\left[\frac{ik}{s} - \frac{1}{s^2} - \frac{1}{s}ik\cos\theta\right]\Big|_{s=R}\end{aligned}$$

we have

$$\begin{aligned}\int_{S_R(\mathbf{M})}\eta[G, U_\mathrm{i}]dS_\mathbf{P} &= -\frac{2\pi R^2 e^{ikR+i\varphi}}{4\pi R}\int_0^\pi d\theta\sin\theta e^{ikR\cos\theta}\left[ik - \frac{1}{R} - ik\cos\theta\right] \\ &= \frac{1}{2}Re^{ikR+i\varphi}\int_{-1}^{1}d\xi e^{ikR\xi}\left[ik\xi + \frac{1}{R} - ik\right] \\ &= \frac{1}{2}Re^{ikR+i\varphi}\left[\frac{\xi e^{ikR\xi}}{R} - \frac{e^{ikR\xi}}{ikR^2} + \frac{e^{ikR\xi}}{ikR^2} - \frac{e^{ikR\xi}}{R}\right]\Big|_{\xi=-1}^{\xi=1} \\ &= \frac{1}{2}e^{ikR+i\varphi}\left[e^{ikR} + e^{-ikR} - e^{ikR} + e^{-ikR}\right] = e^{i\varphi}\,,\end{aligned}$$

which proves (B29).

Substitution of (B28) and (B29) into (B27) gives

$$\int_{S_\varepsilon(\mathbf{M})}\eta[G, U_\mathrm{i}]dS_\mathbf{P} = 0\,. \tag{B30}$$

Combining (B30) with (B18) and using $U(\mathbf{P}) \equiv U_\mathrm{i}(\mathbf{P}) + U_\mathrm{r}(\mathbf{P})$ we obtain that (B23) and (B24) are also valid for $U_\mathrm{i}(\mathbf{P}) = e^{iks\cos\theta+i\varphi}$.

Appendix C Behaviour of $S(g)$ at large g

In this section we estimate the behaviour of (61) at large values of the roughness parameter g. We obtain both the principal term of the asymptotic expansion and an estimate of the remainder.

Without loss of generality we can assume that $T^2 = Pg$, where $P = P(g)$ is a positive parameter. Equation (61) is equivalent to

$$S(g) = \int_0^\infty J_0(v_{xy}\tau)\left[e^{-\tau^2/P} + e^{-g(1-\exp(-\tau^2/T^2))} - e^{-\tau^2/P} - e^{-g}\right]\tau d\tau\,. \tag{C1}$$

Because

$$\int_0^\infty J_0(v_{xy}\tau)e^{-\tau^2/P}\tau d\tau = \frac{1}{2}P\exp\left(-\frac{1}{4}v_{xy}^2P\right)$$

we obtain that (C1) takes the form

$$S(g) = \frac{1}{2}P\exp\left(-\frac{1}{4}v_{xy}^2P\right) + H(g) \tag{C2}$$

where

$$H(g) \equiv \int_0^\infty J_0(v_{xy}\tau)\left[e^{-g(1-\exp(-\tau^2/T^2))} - e^{-\tau^2/P} - e^{-g}\right]\tau d\tau\,.$$

We will show that, when $g \to \infty$,

$$|H(g)| \le \frac{\text{const}}{g}P \qquad (\Leftrightarrow H(g) = O(g^{-1})P)\,. \tag{C3}$$

Use $|J_0(t)| \le 1$ to write

$$|H(g)| \le \int_0^\infty \left|e^{-g\left(1-\exp(-\tau^2/Pg)\right)} - e^{-\tau^2/P} - e^{-g}\right|\tau d\tau\,. \tag{C4}$$

In (C4) change the integration variable from τ to $\xi \equiv ge^{-\tau^2/Pg}$, for which we have

$$d\xi = -\frac{2\tau\xi d\tau}{Pg} \quad \Rightarrow \quad \tau d\tau = -\frac{Pg}{2\xi}d\xi$$

and

$$\xi(0) = g, \qquad \xi(\infty) = 0\,.$$

As a result, the relation (C4) is transformed to

$$\begin{aligned}|H(g)| &\le \int_0^g \frac{Pg}{2\xi}\left|e^{-g+\xi} - \left(\frac{\xi}{g}\right)^g - e^{-g}\right|d\xi \\ &\le \int_0^1 \frac{Pg}{2}\frac{e^\xi - 1}{\xi}e^{-g}d\xi + \int_0^1 \frac{Pg}{2\xi}\left(\frac{\xi}{g}\right)^g d\xi \\ &\quad + \int_1^g \frac{Pg}{2\xi}\left(e^{-g+\xi} - \left(\frac{\xi}{g}\right)^g\right)d\xi + \int_1^g \frac{Pge^{-g}}{2\xi}d\xi\,, \end{aligned} \tag{C5}$$

where we assumed that $g > 1$ and have used the fact that, for $\xi \in [0, g]$,

$$e^{\xi/g} \geq e\frac{\xi}{g} \quad \Leftrightarrow \quad e^{\xi} \geq \left(\frac{e\xi}{g}\right)^g \quad \Leftrightarrow \quad e^{-g+\xi} \geq \left(\frac{\xi}{g}\right)^g .$$

Rewrite (C5) as

$$|H(g)| \leq H_1(g) + H_2(g) + H_3(g) + H_4(g) \tag{C6}$$

and estimate each term on the right separately.

$$H_1(g) \equiv \int_0^1 \frac{Pg}{2}\frac{e^{\xi}-1}{\xi}e^{-g}d\xi \leq \frac{Pg}{2}e^{-g}(e-1) \tag{C7}$$

$$H_2(g) \equiv \int_0^1 \frac{Pg\xi^{g-1}}{2g^g}d\xi = \frac{P}{2}g^{-g} \tag{C8}$$

$$\begin{aligned} H_3(g) &\equiv \int_1^g \frac{Pg}{2\xi}e^{-g+\xi}d\xi - \int_1^g \frac{Pg\xi^{g-1}}{2g^g}d\xi \\ &= \int_1^g \frac{Pg}{2\xi}e^{-g+\xi}d\xi - \frac{P}{2}\left(1-g^{-g}\right) = H_3^{\mathrm{a}}(g) + H_3^{\mathrm{b}}(g) . \end{aligned} \tag{C9}$$

Use integration by parts to estimate $H_3^{\mathrm{a}}(g)$:

$$\begin{aligned} H_3^{\mathrm{a}}(g) &= \frac{Pge^{-g}}{2}\left[\left.\frac{e^{\xi}}{\xi}\right|_1^g + \left.\frac{e^{\xi}}{\xi^2}\right|_1^g + 2\int_1^g \frac{e^{\xi}}{\xi^3}d\xi\right] \\ &= \frac{P}{2}\left[1 - ge^{-g+1} + \frac{1}{g} - ge^{-g+1} + 2ge^{-g}\int_1^g \frac{e^{\xi}}{\xi^3}d\xi\right] . \end{aligned} \tag{C10}$$

Next estimate the last term in (C10):

$$\begin{aligned} ge^{-g}\int_1^g \frac{e^{\xi}}{\xi^3}d\xi &\leq ge^{-g}\int_1^{g/2}\frac{e^{g/2}}{\xi^3}d\xi + ge^{-g}\int_{g/2}^g \frac{e^{\xi}}{(g/2)^3}d\xi \\ &= ge^{-g/2}\left(\frac{1}{2}-\frac{2}{g^2}\right) + \frac{8}{g^2}\left(1-e^{-g/2}\right) = O(g^{-2}) . \end{aligned} \tag{C11}$$

Substitution of (C11) into (C10) gives

$$H_3(g) = \frac{P}{2}\left[1 + O(g^{-1})\right] - \frac{P}{2}\left[1 - g^{-g}\right] = O(g^{-1})P . \tag{C12}$$

Estimate the last term in (C6):

$$H_4(g) \equiv \int_1^g \frac{Pge^{-g}}{2\xi}d\xi = \frac{Pge^{-g}}{2}\log g . \tag{C13}$$

Combining (C6), (C7), (C8), (C12) and (C13), and using the fact that

$$ge^{-g}, \quad g^{-g}, \quad g\log(g)e^{-g}$$

vanish faster than $O(g^{-n})$, where n is an arbitrarily chosen number, we obtain (C3). Substitute (C3) into (C2) and use the fact that $P = T^2/g$ to obtain

$$S(g) = \frac{T^2}{2g}\left[\exp\left(-\frac{v_{xy}^2T^2}{4g}\right) + O(g^{-1})\right] \tag{C14}$$

as $g \to \infty$.

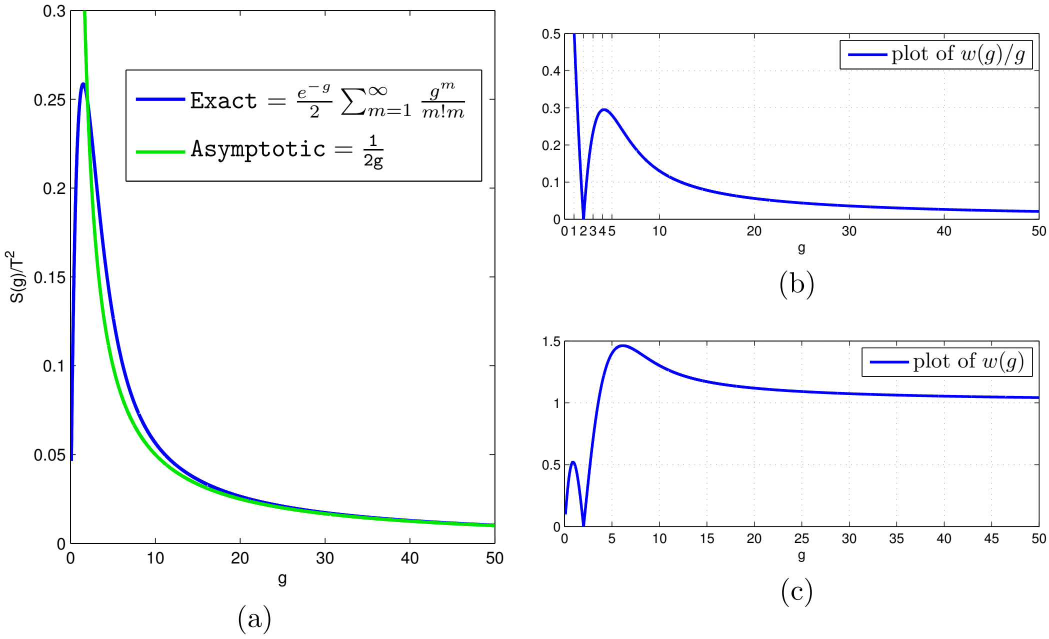

Figure C1: *Comparison of the exact and asymptotic forms of $S(g)$ for the specular case and plots of the remainder $w(g)/g$ (b) and its coefficient $w(g)$ (c).*

Note 20. The remainder $O(g^{-1})$ in (C14) is

$$|O(g^{-1})| = w(g)/g, \tag{C15}$$

where $w(g)$ is a bounded function. We have verified, using (C6) and numeric evaluation of (C7)–(C11) and (C13), that $|w(g)| < 6.2$ for all $g \geq 1$, $|w(g)|$ becoming less than 1.1 when $g > 200$. These are quite coarse estimates, in reality $w(g)$ can be smaller. A more accurate estimate for $w(g)$ is obtained by considering

$$w(g) \equiv g\left|\frac{2g}{T^2}S(g) - \exp\left(-\frac{v_{xy}^2T^2}{4g}\right)\right|,$$

where $S(g)$ is evaluated numerically using either (61) or

$$S(g) = \frac{T^2e^{-g}}{2}\sum_{m=1}^{\infty}\frac{g^m\exp\left(-\frac{v_{xy}^2T^2}{4m}\right)}{m!m}$$

(see [6, eqn. (35), p. 86]). Figure C1 compares the exact and asymptotic forms of $S(g)$ for the specular case. Figure C1 (a) shows that $S(g)$ and its asymptotic approximation are quite close even at moderately small (≈ 1) values of g. The plot of the remainder (C15) in Figure C1 (b) shows that it does not exceed 0.5 at $g \geq 1$. Figure C1 (c) confirms our upper estimates for $w(g)$ and shows that for the specular case $w(g) < 1.5$ if $g \geq 1$ and $w(g) < 1.05$ if $g \geq 50$.

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19. ABSTRACT

The purpose of this document is to provide a detailed and self-contained derivation of the high frequency component SL_1 of the acoustic surface loss formula for a rough interface of impedance mismatch. We thoroughly examine the existing derivations of SL_1, point at their flaws and suggest the corrected form. Much of the report is a technical review in which we critically revisit the fundamentals behind the surface loss formula and cover such aspects as the basics of the Kirchhoff theory, random surfaces and wave scattering from them, large roughness approximation of the diffuse field, and energy flux calculations.

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